

Derivative Pricing

Asset: ^{자산}주식, 채권, 부동산, 금, 현물, 곡물, ...

. 회사의 리스크, 채무 등 실질적인 가치가 있는 '자산'

시간에 따라 가격이 변함, 모든 것의 미래 불가능 (불확실성)

→ 수학적 모형으로 확률과정으로 표현됨

$S^1, S^2, \dots$ 등등

시간 t 에서의 가격: S_t^i

Portfolio: 거래의 대상이 되는 자산을 각각
몇 개씩 가지고 있음.

이론적으로는 미시적, ^(short) 거시적 접근

수학적 모형으로 $\mathbb{R}^N$ 의 원소.

Portfolio $h = (x_1, \dots, x_N) \in \mathbb{R}^N$ 의 시간 t 에서의 가격:

$$V_t^h = \sum_{i=1}^N x_i S_t^i$$

Def An arbitrage is a portfolio $h \in \mathbb{R}^2$ s.t.

① $V_0^h = 0$

② $V_T^h \geq 0$ with probability 1

③ $V_T^h > 0$ with positive probability

Efficient Market Hypothesis:

There is no free-lunch (arbitrage)

Def A financial derivative is an asset whose value is determined by another group of assets guaranteed by the issuer.

Mathematically, A function

$$D_t = f(S_{0 \leq t \leq T}, \dots, S_{0 \leq t \leq T}^n; t)$$

An option is a financial derivative which you can exercise before at maturity to receive a predetermined payoff.

Fix Stock S
(European)

Put option: payoff $(K - S_T)_+$ at maturity T

Call option: payoff $(S_T - K)_+$ at maturity T

K : strike price

Q: What is the 'fair' price of all call/put option before maturity?

Assumption , Deposit/Borrow money at bank (interest rate r)
' Buy/Sell corresponding stock.

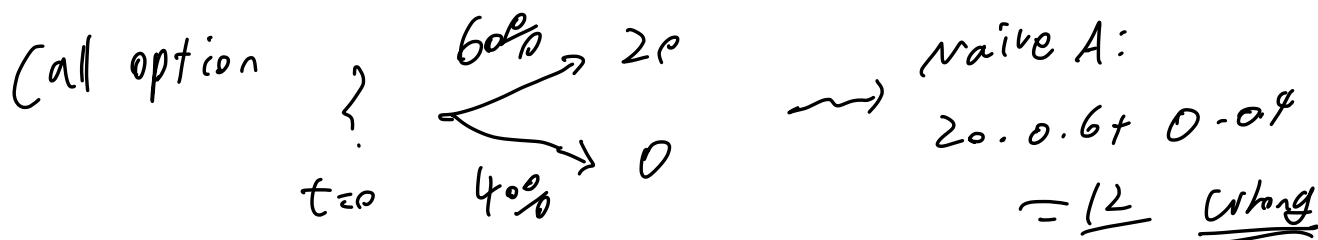
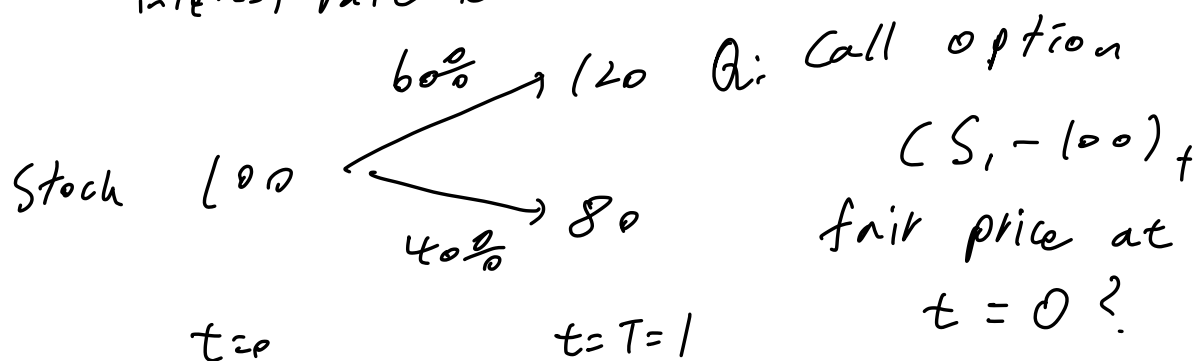
portfolio

$$h = (x, \varphi) \leadsto V_t^h = \underset{\substack{\uparrow \\ \text{bank}}}{x} G_t + \underset{\substack{\uparrow \\ \text{stock}}}{\varphi} S_t$$

' No arbitrage.

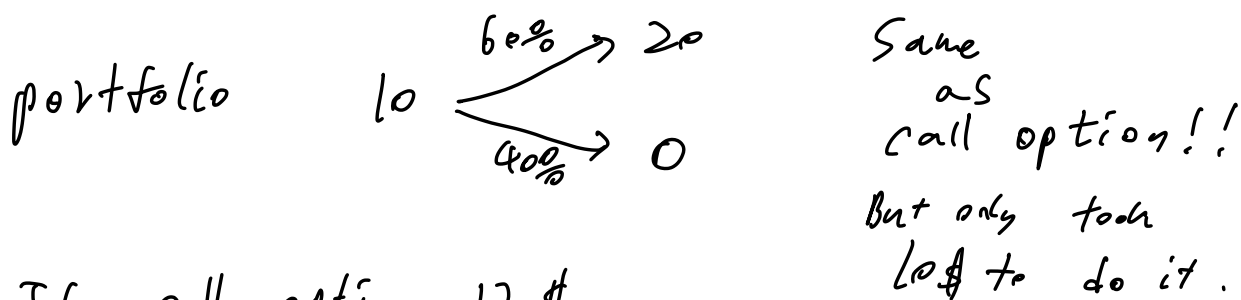
Example

interest rate 0



Borrow 40 from bank, use 10 to buy 0.5 stock

$h = (-40, 0.5)$ ← called 'hedging portfolio'

$$\rightarrow V_t^h = -40 + 0.5 S_t$$


If call option 12\$

Sell a call option - and use 10 to make this portfolio

$\rightarrow$ Free lunch of 2\$

$\rightarrow$ arbitrage!!

A: 12\$

Def A Risk neutral measure $\mathbb{Q}$ is a probability measure such that $\frac{S}{G}$ is a martingale and $\mathbb{P} \sim \mathbb{Q}$.

Thm No arbitrage $\Leftrightarrow$ A Risk neutral measure $\mathbb{Q}$ exists.

Assume

$$dS_t = S_t(\mu dt + \sigma dB_t)$$

$$dG_t = r G_t dt$$

$$\Leftrightarrow S_t = S_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma B_t\right)$$

$$G_t = e^{rt}$$

Self-financing portfolio: $(\pi_t, \phi_t)_{0 \leq t \leq T}$

$$V_t^h = \pi_t G_t + \phi_t S_t$$

$$dV_t^h = \pi_t dG_t + \phi_t dS_t$$

Note

$$\begin{aligned} d\left(\frac{V_t^h}{G_t}\right) &= d\left(V_t^h \cdot \frac{1}{G_t}\right) \quad \text{Ito formula} \\ &= \frac{1}{G_t} dV_t^h + \frac{V_t^h}{G_t^2} d\left(\frac{1}{G_t}\right) + \underbrace{dV_t^h d\left(\frac{1}{G_t}\right)}_0 \\ &= \frac{1}{G_t} (\pi_t dG_t + \phi_t dS_t) + (\pi_t G_t + \phi_t S_t) d\left(\frac{1}{G_t}\right) \\ &= \phi_t \left(\frac{1}{G_t} dS_t + S_t d\left(\frac{1}{G_t}\right) \right) + \underbrace{\pi_t \left(\frac{1}{G_t} dG_t + G_t d\left(\frac{1}{G_t}\right) \right)}_0 \\ &= \phi_t \left(\frac{dS_t}{G_t} \right) \end{aligned}$$

$$\therefore V_t^h = x G_t + G_t \int_0^t \phi_u d\left(\frac{S_u}{G_u}\right)$$

$$\pi_t = \frac{V_t - \phi_t S_t}{G_t}$$

Pricing an option

let V be the price of the option,

assume $V_t = f(t, S_t)$

for $f \in C^{1,2}$, $\sup_t |f(t, x)| \leq C(|x|^p + 1)$

Consider ^{self-financing} portfolio

G, V, S

$$\pi_t, \phi_t^V \equiv 1, \phi_t^S = -f_x(t, S_t)$$

$$\hat{V}_t^h = \phi_t^V V_t + \phi_t^S S_t + \pi_t G_t$$

$$= V_t - f_x(t, S_t) S_t + \pi_t G_t$$

$$d\hat{V}_t^h = \phi_t^V dV_t + \phi_t^S dS_t + \pi_t dG_t$$

$$= \underbrace{dV_t - f_x(t, S_t) S_t}_{\text{Ito}} + \pi_t dG_t$$

$$\left(\begin{aligned} \text{Ito formula } df(t, S_t) &= f_t dt + f_x dS_t + \frac{1}{2} f_{xx} d\langle S \rangle_t \\ &= \left(f_t + \frac{1}{2} \sigma^2 S_t^2 f_{xx} \right) dt + f_x dS_t \end{aligned} \right)$$

$$= \left(f_t + \frac{1}{2} \sigma^2 S_t^2 f_{xx} \right) dt + \pi_t G_t dt$$

(lemma) dV_t^h no dB-term

$$\Rightarrow V_t^h = \int_0^t V_s^h ds$$

proof) $V_t^h = V_0^h G_t + G_t \int_0^t \underbrace{\phi_u}_{=0} d\left(\frac{S_u}{G_u}\right) = V_0^h G_t$

$$\Rightarrow dV_t^h = V_0^h dG_t = r V_0^h G_t dt = r V_t^h dt$$

$$\therefore f_t + \frac{1}{2} \sigma^2 S_t^2 f_{xx} + r \pi_t G_t = r \hat{V}_t$$

$$= r \underbrace{(V_t - f_x S_t + \pi_t G_t)}_{f''}$$

$$\Rightarrow \left\{ \begin{array}{l} f_t + r S_t f_x + \frac{1}{2} \sigma^2 S_t^2 f_{xx} - r f = 0 \end{array} \right.$$

$$f(T, x) = \underbrace{\Phi}_{\text{payoff}}(x)$$

→ Black-Scholes PDE

How to solve BS PDE?

Thm (Feynman-Kac Formula)

$X_u^{t,x}$ sol. of SDE

$$\begin{cases} dX_u = b(u, X_u) du + \sigma(u, X_u) dB_u \\ X_t = x \end{cases}$$

calculate this to solve PDE?

$$f(t, x) = E \left[e^{-\int_t^T r(u, X_u^{t,x}) du} g(X_T^{t,x}) \right]$$

$\Rightarrow$ sol. of

$$\begin{cases} f_t + \frac{1}{2} \sigma^2(t, x) f_{xx} + b(t, x) f_x - r(t, x) f = 0 \\ f(T, x) = g(x) \end{cases}$$

Connection to Schrödinger's Eq.

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi$$

Wick rotation
→
($t \rightarrow i\tau$)

$$\frac{\partial u}{\partial \tau} = \frac{\hbar^2}{2m} \nabla^2 u - Vu$$

Feynman-Kac with $dX_t = \sqrt{\frac{\hbar^2}{m}} dB_t$

$$\leadsto u(x, t) = \mathbb{E} \left[e^{-\int_0^t V(X_s) ds} f(X_t) \mid X_0 = x \right]$$