Derivative Pricing

Asset: 은행 예금, 주식, 채권, 부동산, 금, 현금, 곡물, ...

. 회사의 지분, 채무 등 실질적인 가치가 있는 '자산'

시간에 따라 가격이 변함, 완벽한 예측 불가능 (랜덤성)

⇝ 수학적으로 확률과정으로 표현됨

S^1, S^2, ... 등등

시간 t에서의 가격: S^n_t

Portfolio: 거래의 대상이 되는 자산을 각각 몇개씩 가지고 있는지.

이론적으로는 비정수, 음수도 (short) 허용

수학적으로 $\mathbb{R}^N$의 원소.

Portfolio $h=(x_1, \dots, x_N) \in \mathbb{R}^N$ 의 시간 t 에서의 가치:

$$V^h_t = \sum_{t=1}^{N} x_i S^i_t$$

<u>Def</u> An <u>arbitrage</u> is a portfolio $h \in \mathbb{R}^2$ s.t.

① $V^h_0 = 0$

② $V^h_T \geq 0$ with probability 1

③ $V^h_T > 0$ with positive probability

Efficient Market Hypothesis:

There is no free-lunch (arbitrage)

Def A <u>financial derivative</u> is an asset whose value is determined by another group of assets guaranteed by the issuer.

Mathematically, A function

$$D_t = f(S^1_{0\le s\le t}, \dots, S^n_{0\le s\le t} ; t)$$

An <u>option</u> is a financial derivative which you can exercise before / at maturity to recieve a predetermined payoff.

Fix stock S

(European)
Put option : payoff $(K-S_T)_+$ at maturity T

Call option : payoff $(S_T-K)_+$ at maturity T

K: strike price

Q: What is the 'fair' price of all call/put option before maturity?

<u>Assumption</u>
- Deposit / Borrow money at bank (interest rate r)
- Buy / Sell corresponding stock.

portfolio $h=(x,y) \rightsquigarrow$ $\cdot$ $V^h_t = x G_t + y S_t$ (G_t: bank, S_t: stock)

- No arbitrage.

Example

interest rate 0

$$\begin{tikzcd}[column sep=large] & 120 \\ \text{Stock} \quad 100 \arrow[ur, "60\%"] \arrow[dr, "40\%"'] & \\ & 80 \end{tikzcd}$$

$t=0$ $\qquad\qquad$ $t=T=1$

Q: Call option $(S_1 - 100)_+$ fair price at $t=0$?

$$\begin{tikzcd}[column sep=large] & 20 \\ \text{Call option} \quad ? \arrow[ur, "60\%"] \arrow[dr, "40\%"'] & \\ & 0 \end{tikzcd}$$

$t=0$

$\rightsquigarrow$ Naive A: $20 \cdot 0.6 + 0 \cdot 0.4 = \underline{12}$ wrong

Borrow 40 from bank, use 10 to buy 0.5 stock $\quad h=(-40, 0.5)$ ← called 'hedging portfolio'

$\rightsquigarrow \quad V_t^h = -40 + 0.5 S_t$

$$\begin{tikzcd}[column sep=large] & 20 \\ \text{portfolio} \quad 10 \arrow[ur, "60\%"] \arrow[dr, "40\%"'] & \\ & 0 \end{tikzcd}$$

Same as call option!!

But only took 10\$ to do it.

If call option 12\$

Sell a call option and use 10 to make this portfolio

→ Free lunch of 2\$

→ arbitrage!!

A: 12\$

Def A Risk neutral measure $\mathbb{Q}$ is a probability measure such that $\frac{S}{G}$ is a martingale and $\mathbb{P} \simeq \mathbb{Q}$.

Thm No arbitrage $\iff$ A Risk neutral measure $\mathbb{Q}$ exists.

Assume

$$dS_t = S_t(\mu dt + \sigma dB_t)$$

$$dG_t = r G_t dt$$

$$\Leftrightarrow \quad S_t = S_0 \exp\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma B_t\right)$$

$$G_t = e^{rt}$$

Self-financing portfolio : $(\pi_t, \phi_t)_{0 \le t \le T}$

$$V_t^h = \pi_t G_t + \phi_t S_t$$

$$dV_t^h = \pi_t dG_t + \phi_t dS_t$$

Note

$$d\left(\frac{V_t^h}{G_t}\right) = d\left(V_t^h \cdot \frac{1}{G_t}\right)$$

Ito formula

$$= \frac{1}{G_t} dV_t^h + \frac{V_t^h}{G_t^2} d\left(\frac{1}{G_t}\right) + \underbrace{\overset{(\,)dt + (\,)dB_t}{dV_t^h}\ \overset{((\,)dt)}{d\left(\frac{1}{G_t}\right)}}_{0}$$

$$= \frac{1}{G_t}(\pi_t dG_t + \phi_t dS_t) + (\pi_t G_t + \phi_t S_t)\, d\left(\frac{1}{G_t}\right)$$

$$= \phi_t\left(\frac{1}{G_t} dS_t + S_t d\left(\frac{1}{G_t}\right)\right) + \pi_t \underbrace{\left(\frac{1}{G_t} dG_t + G_t d\left(\frac{1}{G_t}\right)\right)}_{0}$$

$$= \phi_t \left(\frac{dS_t}{G_t}\right)$$

$$\therefore\ V_t^h = x G_t + G_t \int_0^t \phi_u \, d\left(\frac{S_u}{G_u}\right)$$

$$\pi_t = \frac{V_t - \phi_t S_t}{G_t}$$

<u>Pricing an option</u>

let V be the price of the option,

assume $V_t = f(t, S_t)$

for $f \in C^{1,2}$, $\sup_t |f(t,x)| \leq C(|x|^p + 1)$

Consider self-financing portifolio

$$G,\ V,\ S$$

$$\pi_t,\ \phi_t^V \equiv 1,\ \phi_t^S = -f_x(t, S_t)$$

$$\hat{V}_t^h = \phi_t^V V_t + \phi_t^S S_t + \pi_t G_t$$

$$= V_t - f_x(t, S_t) S_t + \pi_t G_t$$

$$d\hat{V}_t^h = \phi_t^V dV_t + \phi_t^S dS_t + \pi_t dG_t$$

$$= \underbrace{dV_t}_{\substack{\shortparallel \\ df(t,S_t)}} - f_x(t, S_t) S_t + \pi_t dG_t$$

$$\left(\underset{\text{Ito formula}}{=} f_t\, dt + f_x\, dS_t + \frac{1}{2} f_{xx}\, \underbrace{d\langle S\rangle_t}_{\substack{\shortparallel \\ \sigma^2 S_t^2 dt}} \right.$$

$$\left. = \left(f_t + \frac{1}{2}\sigma^2 S_t^2 f_{xx}\right) dt + f_x\, dS_t \right)$$

$$= \left(f_t + \frac{1}{2}\sigma^2 S_t^2 f_{xx}\right) dt + r\pi_t G_t\, dt$$

(**Lemma** dV_t^h no dB-term

$$\Rightarrow V_t^h = rV_t^h\,dt$$

proof) $V_t^h = V_0^h G_t + G_t \int_0^t \underbrace{\phi_u}_{=0}\, d\left(\frac{S_u}{G_u}\right) = V_0^h G_t$

$$\Rightarrow dV_t^h = V_0^h\, dG_t = rV_0^h G_t\, dt = rV_t^h\, dt \quad)$$

$$\therefore\ f_t + \frac{1}{2}\sigma^2 S_t^2 f_{xx} + r\pi_t G_t = r\hat{V}_t$$

$$= r(\underbrace{V_t}_{f} - f_x S_t + \pi_t G_t)$$

$$\Rightarrow \begin{cases} f_t + rS_t f_x + \frac{1}{2}\sigma^2 S_t^2 f_{xx} - rf = 0 \\ f(T,x) = \Phi(x) \end{cases} \quad \rightarrow \text{Black-Scholes PDE}$$

Φ: payoff.

How to solve BS PDE?

Thm (Feynman-Kac Formula)

$X_u^{t,x}$ sol. of SDE

$$\begin{cases} dX_u = b(u, X_u)\,du + \sigma(u, X_u)\,dB_u \\ X_t = x \end{cases}$$

calculate this to solve PDE!!

$$f(t,x) = E\left[e^{-\int_t^T r(u, X_u^{t,x})\,du}\, g(X_T^{t,x})\right]$$

$\Rightarrow$ sol. of

$$\begin{cases} f_t + \frac{1}{2}\sigma^2(t,x) f_{xx} + b(t,x) f_x - r(t,x) f = 0 \\ f(T,x) = g(x). \end{cases}$$

Connection to Schrödinger's Eq.

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi$$

Wick rotation $\longrightarrow$ $(t \to i\tau)$

$$\frac{\partial u}{\partial \tau} = \frac{\hbar^2}{2m} \nabla^2 u - Vu$$

Feynman-Kac with $dX_t = \sqrt{\frac{\hbar^2}{m}}\, dB_t$

$$\rightsquigarrow \quad u(x,t) = \mathbb{E}\left[e^{-\int_0^t V(X_s)\,ds} f(X_t) \,\middle|\, X_0 = x \right].$$