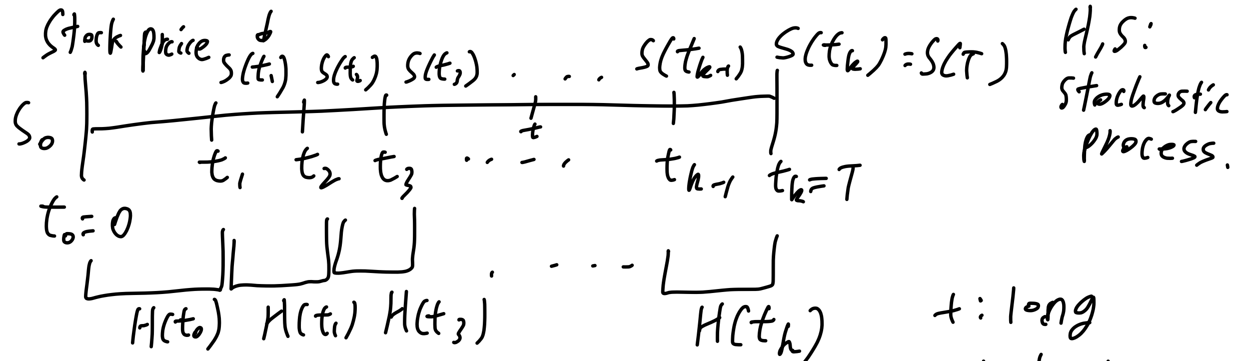
1. Stochastic Integration



H, S: stochastic process.

$+$: long
$-$: short

$$\text{total gain} = H(t_0)(S(t_1) - S(t_0)) + H(t_1)(S(t_2) - S(t_1)) + \cdots + H(t_k)(S(t_k) - S(t_{k-1}))$$

$$= \underbrace{\sum_{n=0}^{k-1} H(t_n)(S(t_{n+1}) - S(t_n))}_{\text{sum of random variable}} \leftarrow \text{random variable}$$

$\underline{\lim |t_{i+1} - t_i| \to 0}$

$$\Rightarrow \text{total gain} = \underbrace{\int_0^T H(t)\, dS(t)}_{\text{stochastic integration}} \leftarrow \text{r.v.}$$

$$\underbrace{G_t = \int_0^t H(u)\, dS(u)}_{\text{Stochastic process}} \quad (0 \le t \le T).$$

In particular

Stochastic process

$$X_t = x + \int_0^t b_u\, du + \int_0^t \sigma_u\, dW_u \leftarrow \text{B.M.}$$

is called a Itô process.

the relation

function

$$X_t = x + \int_0^t b_u(X_u)\, du + \int_0^t \sigma_u(X_u)\, dW_u$$

stochastic process

is called a SDE (stochastic differential eq.)

$$\begin{cases} dX_t = b_t(X_t)\,dt + \sigma_t(X_t)\,dW_t. \\ X_0 = x \end{cases}$$

If σ_u deterministic process i.e. $\sigma_u(\omega) = \sigma_u$,

$$\int_0^t \sigma_u \, dB_u \sim N\left(0, \mathbb{E}\left[\left(\int_0^t \sigma_u \, dB_u\right)^2\right]\right) \quad \forall t.$$

$\lim \sum \sigma_u \Delta B_u$ $\sim N$

σ_u de~~terministic~~

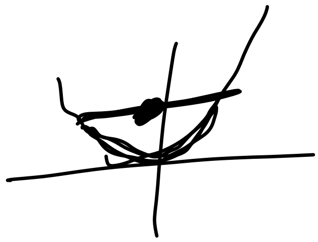
$\int_0^t \sigma_u \, dM_u$ (M: martingale) is a martingale.

2. Quadratic variation.

Let M be a martingale.

Q: Is M^2 a martigale?

A: Not always.



Jensen's inequality:

$$\mathbb{E}[M_t^2 \mid \mathcal{F}_s] \geq \Big(\underbrace{\mathbb{E}[M_t \mid \mathcal{F}_s]}_{M_s}\Big)^2 = M_s^2$$

M^2 'bigger' than martingale.

Def Quadratic variation of a martingale M is the unique increasing stochastic process $\langle M \rangle$ such that $M^2 - \langle M \rangle$ is a martingale.

M^2 'bigger' than martingale.

How much bigger?

$M_{t+dt} = M_t + dM_t$ If M^2 martingale. $(M_t + dM_t)^2 \simeq M_t^2$

$(M_t + dM_t)^2 - M_t^2 = \underbrace{2M_t\, dM_t}_{\text{평균} \simeq 0} + \boxed{(dM_t)^2}$

↑ This much bigger.

$$\langle M \rangle_t = \int_0^t (dM_u)^2 = \lim \sum (M_{t_{i+1}} - M_{t_i})^2$$

$$d\langle M \rangle_t = (dM_u)^2$$

Def More generally if X is a semimartingale. (Itô process ⊂)

it can be written as $X_t = \boxed{A_t} + \boxed{M_t}$ (M_t: mg.)

$$\langle X \rangle_t := \langle M \rangle_t .$$

Examples. B: B.M.

(i) $\mathbb{E}[B_t^2 - t \mid \mathcal{F}_s]$

$= \mathbb{E}[(B_t - B_s + B_s)^2 - t \mid \mathcal{F}_s]$

$= \underbrace{\mathbb{E}[(B_t - B_s)^2 \mid \mathcal{F}_s]}_{t} + 2\,\mathbb{E}[(B_t - B_s)\mathcal{F}_s]$ (독립, 독립.)

$B_t - B_s \sim N(0, t-s)$ $\mathbb{E}[B_s^2 \mid \mathcal{F}_s] - t.$ (measurable)

$= (t-s) + 2\cdot 0 + B_s^2 - t = B_s^2 - s.$

$\Rightarrow B_t^2 - t$: martingale.

$$\boxed{\langle B \rangle_t = t .}$$

$$\lim \sum (B_{t_{i+1}} - B_{t_i})^2 = t.$$

(ii) $$X_t = X_0 + \int_0^t b_u \, du + \underbrace{\boxed{\int_0^t \sigma_u \, dB_u}}_{\text{martingale.}}$$

$$\langle X \rangle = \langle \int_0^t \sigma_u \, dB_u \rangle$$

$$d\langle M \rangle = (dM)^2$$

$$\begin{aligned}(\sigma_t \, dB_t)^2 &= \sigma_t^2 (dB_t)^2 \\ &= \sigma_t^2 \, \underbrace{d\langle B \rangle_t}_{dt} \\ &= \sigma_t^2 \, dt.\end{aligned}$$

$$\boxed{\langle X \rangle_t = \int_0^t \sigma_u^2 \, du.}$$

Def The quadratic covariation of two martingales M, N is the unique process $\langle M, N \rangle$ such that $MN - \langle M, N \rangle$ is a martingale. martingale.

If $X = A_1 + M_1$; $Y = A_2 + M_2$

$$\langle X, Y \rangle = \langle M_1, M_2 \rangle$$

As before,

$$d\langle X, Y \rangle_t = dX_t \, dY_t$$

$$\langle X, Y \rangle_t = \lim \sum (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i}).$$

Examples

(i) $t = t + \underbrace{0}_{\text{martingale}}$, $B_t = 0 + \underbrace{B_t}_{\text{martingale part.}}$

$\Rightarrow \langle t, B_t \rangle = \langle 0, B_t \rangle = 0$

i.e. $dt\, dB_t = 0$

(ii) $dX_t = b_1(t)\, dt + \underline{\sigma_1(t)\, dB_t}$

$dY_t = b_2(t)\, dt + \underline{\sigma_2(t)\, dB_t}$

$$dX_t\, dY_t = \sigma_1(t)\sigma_2(t) \underbrace{(dB_t)^2}_{= d\langle B \rangle_t = dt}$$

$$= \sigma_1(t)\sigma_2(t)\, dt.$$

$$\therefore \langle X, Y \rangle_t = \int_0^t \sigma_1(u)\sigma_2(u)\, du.$$

Itô isometry.

M : martingale.

$$\underbrace{E\left[\left(\int_0^t \sigma_u\, dM_u\right)^2\right]}_{Var\left(\int_0^t \sigma_u\, dM_u\right)} = \underline{E\left[\int_0^t \sigma_u^2\, d\langle M \rangle_u\right]}$$

$$\int_0^t \sigma_u\, dB_u \sim N\left(0, E\left[\int_0^t \sigma_u^2\, dB_u\right]\right)$$

3. Itô Formula.

Recall

$$f(t,x) = f(0,0) + \int \frac{\partial f}{\partial t}(t,x)\,dt + \int \frac{\partial f}{\partial x}(t,x)\,dx.$$

i.e. $df(t,x) = \frac{\partial f}{\partial t}\,dt + \frac{\partial f}{\partial x}\,dx$

: Taylor expansion (up to first order).

$(dt)^2 \to 0$ $\quad dt\,dx \to 0$ $\quad (dx)^2 \to 0$

$$f(t+dt, x+dx) = f(t,x) + \underbrace{\frac{\partial f}{\partial t}\,dt + \frac{\partial f}{\partial x}\,dx}_{1^{st}\text{ order}} + \underbrace{\frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{\partial^2 f}{\partial t \partial x}\,dt\,dx + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dx)^2}_{\text{second order}}$$

$(dt)^2 = 0$, $dt\,dX_t$, $+ \cdots$

$\underline{(dX_t)^2 = d\langle X\rangle_t} \neq 0.$

Similarly.

$$df(t, X_t) = \frac{\partial f}{\partial t}\,dt + \frac{\partial f}{\partial x}\,dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\,d\langle X\rangle_t$$

$$f(t,X_t) = \int_0^t \frac{\partial f}{\partial t}(u,X_u)\,du + \int_0^t \frac{\partial f}{\partial x}(u,X_u)\,du + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(u,X_u)\,d\langle X\rangle_u.$$

$$df(X_t, Y_t) = \frac{\partial f}{\partial x}\,dX_t + \frac{\partial f}{\partial y}\,dY_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\,d\langle X\rangle_t + \frac{\partial^2 f}{\partial x \partial y}\,d\langle X,Y\rangle_t + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}\,d\langle Y\rangle_t.$$