Introduction to Non-commutative Geometry

(based on a talk by Myung Jin Jeong)

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May 9, 2025

Keywords: Noncommutative geometry · dg-category · Morita equivalence · Hochschild (co)homology · derived stacks · compact objects · open/closed string analogy · Motives

Contents

1 Introduction

1.1 Affine spaces

Classical geometry is divided into several schools, and the distinction between these areas is often stated by what are the favorite objects of their practitioners: Topology is the study of topological spaces, differential geometry is that of smooth manifolds, complex geometry is tied to complex manifolds, and algebraic geometry to varieties.

In any case, a geometric object consists of an underlying set and some additional data called *structure sheaf.* Examples are furnished by continuous functions C_X on topological spaces, smooth functions C_X^∞ on smooth manifolds, holomorphic functions $\mathcal{O}_X$ on complex, etc. In fact, it is important that the information captured by a sheaf transcends a single set of functions. For instance, the sheaf of continuous functions on a topological space X is not a single algebra $C(X)$ but a collective assignment $U \mapsto C(U)$ of algebras to the open subsets of X. Moreover, the assignment of function algebras behaves coherently with respect to the inclusion relations between the open sets. For example, whenever one has $X = U \cup V$, the functions $C(X)$ on X can be obtained by 'gluing' the functions in $C(U)$ and $C(V)$ that coincide when restricted to $U \cap V$. More generally, any function is a coherent combination of simpler pieces that live in arbitrarily small open subsets. It is in this sense that sheaves are said to capture the local information about the space.

$$C(U \cap V) \rightrightarrows C(U) \oplus C(V) \longrightarrow C(X) \tag{1}$$

However, within any type of geometry, there are such nice spaces in which the global sections determine every other aspect of the structure sheaf—this is the defining property of *affine spaces.*

Example 1.1. *The assignment of the coordinate ring* $\mathbb{C}[x_1, \ldots, x_n]/(p)$ *to the affine algebraic variety in* $\mathbb{A}^n$ *determined by* $p(x) = 0$ *establishes an equivalence of affine varieties and finitely generated reduced algebras over* $\mathbb{C}$.

Example 1.2. *A smooth manifold* M *determines a 'smooth algebra'* $C^\infty(M)$*, and vice versa.*

Example 1.3. *Compact hausdorff space* $\leftrightarrow$ *unital* C^* *algebra* $C(X)$

$$\begin{tikzcd} C(X) \arrow[dr] & X \arrow[l] \\ & \operatorname{Specm} C(X) \arrow[u, "\simeq"'] \end{tikzcd} \qquad \begin{tikzcd} \mathscr{A} \arrow[r] & \operatorname{Specm} \mathscr{A} \arrow[dl] \\ C(\operatorname{Specm} \mathscr{A}) \arrow[u, "\simeq"] & \end{tikzcd} \tag{2}$$

Example 1.3 has the following twist: Locally compact Hausdorff spaces produces a (not necessarily unital) C^* algebra $C_c(X)$ of functions with compact support. Note that if X is only locally compact but not compact, the unit function 1 does not have compact support. The inverse construction of $X \mapsto C_c(X)$ is schematically given by

$$A \longmapsto A \oplus \mathbb{C}1 =: A^+ \longmapsto \operatorname{Specm} A^+ \simeq \hat{X} \longmapsto \hat{X} \backslash \{*\} \simeq X \tag{3}$$

where $\hat{X}$ is the one-point compactification of X.

Example 1.4. *Also, for 'nice' measure spaces, the assignment*

$$(X, \sigma, \mu) \longmapsto L^\infty(X, \sigma, \mu) \tag{4}$$

has an inverse construction: Given a commutative von Neumann algebra, we can realize it as the algebra of essentially bounded functions of a measure space (X, σ, μ)*. In particular, the sigma algebra* σ *needs to be specified from the data of a commutative von Neumann algebra.*

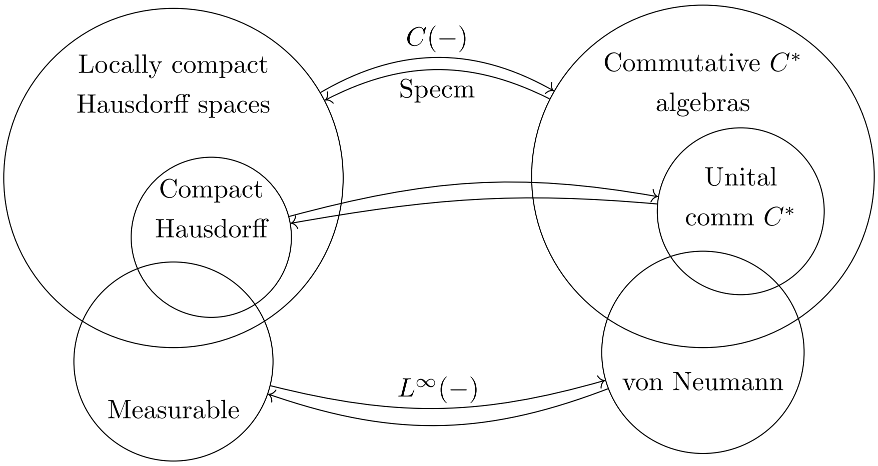

Figure 1: Duality between space and algebra.

Example 1.5. *Stein manifolds are the analogues of affine varieties for the subject of complex manifolds.*

All of these examples involve commutative algebras, and one could even say that every 'nice' commutative algebra arises this way, namely from 'affine spaces'. Conversely, a well-chosen class of commutative algebras characterizes a type of geometry by specifying its affine objects. In particular, every smooth manifold is 'affine'.

Slogan 1.6. *Commutative algebras = affine spaces. Also see theorem 2.19.*

Question 1.7. *How can we deal with non-commutative algebras and non-affine spaces?*

1.2 Non-affine spaces

For the case of non-affine spaces, the algebra of functions globally defined on the space X is inadequate to understand the geometry. What one needs is the understanding of the structure sheaf $\mathcal{O}_X$ in its entirety, which recovers the global function algebra upon taking the global section functor $\mathcal{O}_X \mapsto \Gamma(X, \mathcal{O}_X)$.

In what follows, we will explore how algebro-geometric ideas might enable us to deal with non-affine spaces, not entirely giving up the global viewpoints illustrated in Sec. 1.1.

1.2.1 The Serre-Swan theorem

We remarked that the geometry of a general, non-affine space X is encoded by the structure sheaf $\mathcal{O}_X$, which is a sheaf of rings over the space. Study of a sheaf of rings is naturally connected to the study of the category $\mathcal{O}_X$-Mod of $\mathcal{O}_X$-modules, which is canonically determined by $\mathcal{O}_X$. This is analogous to the situation in ring theory, where understanding a ring R is often translated into understanding the category ${}_R\mathsf{Mod}$ of R-modules. Roughly speaking,

$$\text{Ring } R : {}_R\mathsf{Mod} = \text{Structure sheaf } \mathcal{O}_X : \mathcal{O}_X\text{-}\mathsf{Mod} \tag{5}$$

More geometrically, however, it often suffices to study a special class of $\mathcal{O}_X$-modules called *coherent sheaves*, whose category is a full subcategory of $\mathcal{O}_X\text{-}\mathsf{Mod}$ and denoted by Coh_X. It is a 'nice' category that generalizes the category of (finite rank) vector bundles"

Example 1.8. *In algebraic geometry, the coherent sheaves over the affine scheme $X = \operatorname{Spec} R$, where $R \in \mathsf{CommRing}$, are precisely the finitely generated R-modules: $\mathsf{Coh}(\operatorname{Spec} R) = {}_R\mathsf{mod}$.*

More generally, we can allow for the generalization of infinite rank vector bundles to form a full subcategory QCoh_X of $\mathcal{O}_X$-modules, called *quasi-coherent sheaves.* For an affine scheme $X = \operatorname{Spec} R$, we have $\mathsf{QCoh}(\operatorname{Spec} R) = {}_R\mathsf{Mod}$.

The subcategory $\mathsf{Vect}(X)$ of vector bundles is also of much interest.

Definition 1.9 (Global section functor)**.** *Let X be a smooth manifold, and define the functor*

$$\Gamma : \begin{array}{rcl} \mathsf{Vect}(X) & \to & {}_A\mathsf{Mod} \\ E & \mapsto & \Gamma(E) \end{array} \tag{6}$$

where $A = C^\infty(X)$.

Observation 1.10. *Γ lands in the subcategory ${}_A\mathsf{proj}$ of finitely generated projective modules.*

Theorem 1.11 (Serre–Swan)**.** *Let X be either smooth manifold or smooth algebraic variety. The global section functor gives an equivalence between categories $\mathsf{Vect}(X)$ and ${}_A\mathsf{proj}$.*

The Serre–Swan theorem was first established in an algebraic setting, but soon extended to topological spaces, real smooth manifolds, Stein manifolds, etc.

2 First step toward non-commutative geometry

In this section, we motivate the following definition.

Definition 2.1. *Let A be an associative algebra which is not necessarily commutative. Then*

- $\mathsf{QCoh}(A)$ *is defined to be the category of finitely generated A-module*
- $\mathsf{Vect}(A)$ *is defined to be the category of finitely generated projective A-module.*

In fact, one must consider $\mathcal{D}(\mathsf{QCoh}(A))$ in place of $\mathsf{QCoh}(A)$, and consider the subcategory of compact objects to get $\mathsf{Vect}(A)$.

Definition 2.2 (Compact object)**.** *An object c in a category C that admits all filtered colimits is called <u>compact</u> if the corepresentable functor $\operatorname{Hom}_{\mathsf{C}}(c, -)$ preserves filtered colimits. That is, given a directed category D and a functor $F : \mathsf{D} \to \mathsf{C}$, the map*

$$\operatorname{colim} \operatorname{Hom}_{\mathsf{C}}(c, F(d)) \to \operatorname{Hom}_{\mathsf{C}}(c, \operatorname{colim} F(d)) \tag{7}$$

is a bijection.

The notion of compact object is a generalization of projective modules, and is called perfect A-modules in our context. That is, a A-module is perfect if and only if it is a compact object of $\mathcal{D}(\mathsf{QCoh}(A)))$.

Specifically, compact objects in $\mathcal{D}(\mathsf{QCoh}(A)))$ can be understood in analogy with finite dimensional vector spaces, which are the compact objects in Vect_k. In the category Vect_k, for any finite dimensional vector space V, the functor $\mathrm{Hom}(V,-)$ is left adjoint to $-\otimes V$:

$$\mathrm{Hom}_k(\mathrm{Hom}_k(V,W),U) \simeq \mathrm{Hom}_k(W, U\otimes V). \tag{8}$$

As a result, V is a compact object by the general principle that left adjoint functors preserve colimits. In particular, it preserves cokernels and thus $\mathrm{Hom}(V,-)$ is a (left and right) exact functor. Also note that compact objects and dualizable objects coincide in this case. We expect that the compact objects in $\mathcal{D}(\mathsf{QCoh}(A)))$ enjoy similar properties.

2.1 Hochschild (co)homology

The point of departure is the differential forms. Given a commutative algebra A, in virtue of Serre-Swan theorem, defining the cotangent bundle T^*X_A is equivalent to defining the space of 1-forms $\Omega^1_A = \Gamma(T^*X_A)$.

Ω^1_A is generated by the “differentials” $a\cdot \mathrm{d}(b)$, where d satisfies Leibniz rule.

$$a\cdot \mathrm{d}(bc) = ab\cdot \mathrm{d}(c) + a\cdot \mathrm{d}(b)c \tag{9}$$

This equips Ω^1_A with a right A-module, hence a A-bimodule structure:

$$\mathrm{d}(b)\cdot c = \mathrm{d}(bc) - b\cdot \mathrm{d}(c) \tag{10}$$

As a naive attempt to represent the form $a\cdot\mathrm{d}(b)$ by an element of an algebra, one tries $a\otimes b\in A\otimes A$. However, we need the following consideration: Since one expects $a\,\mathrm{d}(1)\equiv 0$, not all elements of $A\otimes A$ can be used to represent a nontrivial 1-form. Instead, one sets $\mathrm{d}(a) := 1\otimes a - a\otimes 1$ so that only those elements in $\ker\mu\subset A\otimes A$, where μ is the multiplication map of A.

$$\Omega^1_A \simeq \frac{\ker\mu}{\langle ab\otimes c - a\otimes bc + ca\otimes b\rangle} \in {}_{A\otimes A^{op}}\mathsf{Mod} \tag{11}$$

There is an equivalent description of forms. Consider the multiplication map

$$\mu: \begin{array}{ccc} A\otimes A & \to & A \\ a\otimes b & \mapsto & ab \end{array} \tag{12}$$

whose kernel is generated by the elements $a\otimes 1 - 1\otimes a$. Defining an operator d by $\mathrm{d}a := a\otimes 1 - 1\otimes a$, d satisfies the Leibniz rule:

$$\begin{aligned} \mathrm{d}(ab) &= ab\otimes 1 - 1\otimes ab \\ &= (a\otimes 1 - 1\otimes a)(b\otimes 1) + (1\otimes a)(b\otimes 1 - 1\otimes b) \\ &= (\mathrm{d}a)b + a(\mathrm{d}b) \end{aligned} \tag{13}$$

Recall that a space X is said to be *smooth* if the tangent sheaf TX is well-defined as a vector bundle. The tangent sheaf is defined to be the dual of the cotangent sheaf, so it suffices to check that the cotangent sheaf T^*X is a vector bundle. By the Serre–Swan theorem, this holds if and only if $\Gamma(T^*X) = \Omega^1_A$ is finitely generated projective. This leads us to the following definition:

Definition 2.3. *A commutative associative algebra A is smooth if $\ker \mu$ is a finitely generated projective A-module.*

A smooth space X has differential forms and vector fields. As we defined the 1-forms as the kernel of the multiplication map, how can we obtain higher forms and vectors? To this end, we extend the map μ to $A^{\otimes n}$ through the Leibniz rule. Interestingly, the modules $A^{\otimes n}$ collectively form a chain complex:

Definition 2.4 (Bar complex)**.** *On the complex $\bigoplus_{n\geq 0} A^{\otimes(n+2)}[-n]$ of $A \otimes A^{op}$-modules,*

$$\begin{tikzcd} \cdots \arrow[r] & A^{\otimes 4} \arrow[r] & A^{\otimes 3} \arrow[r] & A^{\otimes 2} \arrow[d, "\mu"] \\ & & & A \end{tikzcd} \tag{14}$$

where $A^{\otimes(n+2)} = A_{\text{first}} \otimes A^{\otimes n} \otimes A_{\text{last}}$ is naturally a $A \otimes A^{op}$-module by acting on $A_{\text{first}} \otimes A_{\text{last}}$, define a differential $b' : A^{\otimes(\bullet+1)} \to A^{\otimes \bullet}$ by setting

$$b'(a_0 \otimes a_1 \otimes \cdots \otimes a_n) := \sum_{i=0}^{n-1} (-1)^i a_0 \cdots a_{i-1} \mu(a_i \otimes a_{i+1}) a_{i+2} \cdots a_n \tag{15}$$

on $A^{\otimes(n+1)}$. Then $b'^2 = 0$, and moreover, $H_*(A^{\otimes \bullet}, b') = 0$ for $* > 0$. The resulting chain complex is called the bar resolution of A.

Note 2.5. *The bar complex is a free resolution of A as an A-bimodule.*

Thus, as $A^{\otimes \bullet}$ is an acyclic resolution of the multiplication map $\mu : A \otimes A \to A$, one can extract higher (co)homology data of μ by applying a functor to the complex $A^{\otimes \bullet}$ and then taking the homology. Also note that the category we want to work in is that of $A \otimes A^{op}$-modules. This is in part motivated by the fact that the Leibniz rule automatically endows a bimodule structure on forms.

Applying the functor $A \otimes -$ or $\mathrm{Hom}(-, A)$ and then taking homology, we get the Hochschild (co)homology:

Definition 2.6. *For an associate algebra A, the Hochschild homology $HH_*(A)$ and $HH^*(A)$ is defined by*

$$HH_*(A) := \mathrm{Tor}_*^{A \otimes A^{op}}(A, A), \quad HH^*(A) := \mathrm{Ext}^*_{A \otimes A^{op}}(A, A). \tag{16}$$

Specifically, the torsion groups are computed by tensoring the bar complex with A as a A-bimodule. In the resulting complex, we have $n + 1$ copies of A in degree n. Its cohomology is, in degree 1,

$$HH_1(A) = \frac{\{a \otimes b\} \in \ker b_1}{\langle ab \otimes c - a \otimes bc + ca \otimes b \rangle}. \tag{17}$$

Question 2.7. *$HH_n(A) \simeq \Omega^n_A$?*

Answer: Yes, this is a theorem of Hochschild–Kostant–Rosenberg.

Theorem 2.8 (Hochschild–Kostant–Rosenberg)**.** *If k is a field and A a commutative k-algebra which is finitely presented and smooth over k, then there are isomorphisms of graded k-algebras*

$$HH_\bullet(A) \xrightarrow{\sim} \Omega^\bullet_{A/k}, \quad HH^\bullet(A) \xleftarrow{\sim} \textstyle\bigwedge^\bullet_A \mathrm{Der}_k(A, A). \tag{18}$$

So, the Hochschild homology is the differential form, and dually, the Hochschild cohomology gives multivector fields.

Remark 2.9. *The multiplicative structure exists only for commutative case (where the multiplication is the wedge product). In the non-commutative case, we don't expect that the differential forms constitute an algebra. In the commutative case, Ω^n is essentially a free envelope over Ω^1.*

Remark 2.10. *The zeroth Hochschild cohomology turns out to be the center $Z(A)$, not A! so the degree-0 piece should not be thought of as the base function ring — it is an accidental feature of the commutative setting. In parallel, the zeroth Hochschild homology produces the abelianization $A/[A, A]$ of A.*

We promote Thm. 2.8 to a definition for non-commutative setting.

Definition 2.11. *Vector fields = Hochschild cohomology, forms = Hochschild homology.*

Remark 2.12. *Hochschild homology behaves functorially, while Hochschild cohomology does not. Given an algebra map $A \to B$ or a functor $C \to D$ between categories, it produces a vector space map on Hochschild homology. However, it is not the case for Hochschild cohomology in general. This is already anticipated in the commutative case: Forms can always be pulled back, while vector fields are not well-behaved under pushforward. Algebraically, Hochschild homology is obtained from tensor product, which canonically induces a new morphism from the old one $M \to N$. In contrast, Hom functor does not naturally induce a new morphism.*

In any case, forms and vectors *must* behave well under isomorphisms.

Question 2.13. *Are these well-defined up to isomorphisms?*

2.2 Theoretical sophistication

2.2.1 Morita equivalence

Definition 2.14. *Two algebras A, B are Morita equivalent if ${}_A\mathsf{Mod} \simeq {}_B\mathsf{Mod}$.*

It is known that two algebras a,b are Morita equivalent iff there is a (A, B)-bimodule M and (B, A)-bimodule N such that

$$M \otimes_B N \simeq A, \qquad N \otimes_A M \simeq B \tag{19}$$

Proposition 2.15. *Our notions are well-defined under Morita equivalence.*

Recall the problem of classifying topological spaces up to homeomorphism. This is intractable, so instead we consider weak homotopy equivalence. Analogously, we blur the boundary between different spaces by considering Morita equivalent classes. Morita theory is the "homotopy theory" that makes Morita equivalence the weak-equivalence!

Definition 2.16. *An affine space is a category of the form ${}_A\mathsf{Mod}$.*

2.2.2 Derived categories

Motivation from complex geometry

A basic topological data of a smooth manifold is its de Rham cohomology. In this brief subsection, we investigate which geometric structures could we use to extract de Rham cohomology, in the case of compact complex manifolds:

$$X : \text{Compact complex manifold} \quad \rightsquigarrow \quad H^n_{\mathrm{dR}}(X) := H^n(\Omega^\bullet(X)) \tag{20}$$

In the holomorphic setting, which is a natural starting point of complex geometry, we may guess that

$$H^n_{\mathrm{dR}}(X) \overset{?}{=} H^n(\Omega^\bullet_{\mathrm{hol}}(X)). \tag{21}$$

But it turns out that this route is not very useful, as there are not many holomorphic functions on a compact complex manifold. To wit: $\mathcal{O}_X(X) \simeq \mathbb{C}$. Instead, we use sheaf cohomology. Consider the chain complex of sheaves

$$\mathcal{O}_X \longrightarrow \Omega^1_{\mathrm{hol},X} \longrightarrow \Omega^2_{\mathrm{hol},X} \longrightarrow \cdots \tag{22}$$

from which we wish to exact cohomology data. A practical way of doing this is to take the resolution $\Omega^{m,1}_X \xrightarrow{\overline{\partial}} \Omega^{m,2}_X \to \cdots$ of each $\Omega^m_{\mathrm{hol},X} =: \Omega^{m,0}_X$. This leads to the bigraded complex

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
 & \Omega^{0,2}(X) \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"] & \Omega^{1,2}(X) \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"] & \Omega^{2,2}(X) \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"] & \cdots \\
 & \Omega^{0,1}(X) \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"] & \Omega^{1,1}(X) \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"] & \Omega^{2,1}(X) \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"] & \cdots \\
\mathcal{O}_X \arrow[r, equal] & \Omega^{0,0}(X) \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"] & \Omega^{1,0}(X) \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"] & \Omega^{2,0}(X) \arrow[u, "\overline{\partial}"] \arrow[r, "\partial"] & \cdots
\end{tikzcd} \tag{23}$$

Starting from the vertical cohomology of Eq. (23), which is called the *Dolbeault cohomology*, one can compute the total cohomology (= de Rham cohomology) via the Hodge-de Rham (or Frölicher) spectral sequence. In particular, if X is compact Kähler, the spectral sequence collapses at the first page; in this case

$$H^n_{\mathrm{dR}}(X) \simeq \bigoplus_{p+q=n} H^{p,q}_{\overline{\partial}}(X). \tag{24}$$

Lesson: A suitable category to characterize a compact complex manifold is the category of chain complexes of sheaves.

Thus, we move on from ${}_A\mathsf{Mod}$ to $\mathsf{Ch}({}_A\mathsf{Mod})$ Furthermore, we identify two spaces whenever there is a single morphism f between them that induces isomorphism on cohomology. Equivalently, we localize the category $\mathsf{Ch}({}_A\mathsf{Mod})$ at the weak equivalences W. This leads to the derived category $\mathcal{D}({}_A\mathsf{Mod})$.

$$\mathcal{D}({}_A\mathsf{Mod}) := \mathsf{Ch}({}_A\mathsf{Mod})[W^{-1}] \tag{25}$$

Definition 2.17. *An affine space is a derived category $\mathcal{D}^b({}_A\mathsf{Mod})$. A vector bundle is an object of its full subcategory $\mathcal{D}^b({}_A\mathsf{proj})$. The differential forms and multivector fields are $HH_*(A)$ and $HH^*(A)$, respectively.*

There is also a categorical characterization:

$$\text{Vector bundle} = \text{compact object} \tag{26a}$$
$$\text{Hochschild cohomology} = \mathrm{End}(\mathrm{id}_{{}_A\mathsf{Mod}}) \tag{26b}$$

In effect, we can do geometry over a wide variety of categories C considered as a space. To be more specific, we consider *differential graded categories*, where the morphism set between a pair of objects is a chain complex.

Definition 2.18. *A space is a differential graded category C.*

- *A local chart is an embedding ${}_A\mathsf{Mod} \hookrightarrow \mathsf{C}$ for some dg-algebra A*
- *C is affine if there is a global chart.*
- *C is a smooth/proper space if the underlying dg-category is smooth/proper.*
- *A vector bundle over C is a compact object of the underlying dg-category.*
- *Differential forms are the Hochschild homology $HH_*(\mathsf{C})$.*
- *Vector fields are the Hochschild cohomology $HH^*(\mathsf{C})$.*

Theorem 2.19. *The following is true.*

1. *For every nice dg-category C, there is a unique scheme X such that $\mathsf{C} = \mathcal{D}(\mathsf{QCoh}(X))$.*
2. *$\mathsf{C} = \mathcal{D}(\mathsf{QCoh}(X))$ is smooth/proper iff X is smooth/proper.*
3. *For all nice dg-categories C, there is a dg-algebra A such that $\mathsf{C} = \mathcal{D}({}_A\mathsf{Mod})$.*

The third part of Theorem 2.19 says that every space is affine in non-commutative setting. For example, the projective space $\mathbb{P}^n$ is not affine in the usual sense, but Theorem 2.19 suggests that there is a global cover of the category $\mathcal{D}^b(\mathsf{QCoh}(\mathbb{P}^n))$. This is the content of the Beilinson's theorem.

Theorem 2.20 (Beilinson)**.** *The bounded derived category of coherent sheaves on the projective space $\mathbb{P}^n$ is given by*

$$A^{op} := \mathrm{End}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(n)), \tag{27a}$$
$$\mathcal{D}^b(\mathsf{QCoh}(\mathbb{P}^n)) \simeq \mathcal{D}^b({}_A\mathsf{Mod}). \tag{27b}$$

Next, a comment regarding the first part of Theorem 2.19 is in order: Without the "niceness" of dg categories, we cannot identify a corresponding scheme X uniquely. One then resorts to a Morita-theoretic, general non-commutative geometry.

Example 2.21. *The* homological mirror symmetry *establishes an equivalence of categories*

$$\mathcal{D}^b(\mathsf{Tw}^\pi\mathsf{Fuk}(X,\omega)) \simeq \mathcal{D}^b(\mathsf{QCoh}(X^\vee)) \tag{28}$$

where $X^\vee$ is the mirror Calabi-Yau manifold of X. Given the pair (S, E_S) of a subspace and a vector bundle over it, we obtain a pair of categories by a procedure called topological twist:

$$\begin{tikzcd} & (S,E_S) \arrow[dl, "A\text{-}twist"'] \arrow[dr, "B\text{-}twist"] & \\ \mathsf{Tw}^\pi\mathsf{Fuk}(X,\omega) & & \mathsf{QCoh}(X) \end{tikzcd} \tag{29}$$

Then there is a mirror manifold $X^\vee$ and its own A- and B-twist. The mirror symmetry interchanges the twists! When X is a Calabi-Yau threefold, the A- and B-twist are the only possible topological twists. Again, mirror symmetry interchanges them. Morita theory is indispensable in stating the mirror symmetry, since $X^{\vee\vee} \simeq X$ only up to Morita equivalence!

Remark 2.22. *Under a suitable (categorical) characterization of string theory, the non-commutative geometry is equivalent to the open string theory. Specifically, given a category C, one interprets the objects of C as the D-branes and morphisms as the open strings stretched between the branes. There is a canonical way to produce a closed string from open strings, i.e. cyclic concatenation of open strings. Accordingly, one can define the Hochschild complex in the category C by considering a cyclic concatenation of morhpisms:*

$$CH_n(\mathsf{C}) := \prod_{x_0,\dots x_n \in \mathsf{C}} \mathrm{Hom}(x_0,x_1)\times\cdots\times\mathrm{Hom}(x_n,x_0) \tag{30}$$

$$b(f_0,\cdots f_n) = \sum_{i=0}^{n}(-1)^i f_0\cdots(f_i\circ f_{i+1})\cdots f_n \tag{31}$$

In other words, the Hochschild homology is an object of a closed string theory derived from an open string theory.

2.3 Slogan of non-commutative geometry

Our discussions of non-commutative geometry are succinctly summarized by Table 1.

More generally, we can do geometry with arbitrary dg-algebra A, which we regard as the function algebra of a space X. The coherent sheaves over X are $\mathcal{D}(\mathsf{QCoh}(X)) = \mathcal{D}({}_A\mathsf{Mod})$.

Slogan 2.23. *All geometric information of X is encoded in the derived category $\mathcal{D}(\mathsf{Vect}(X))$.*

3 Spectra

3.1 Motivation

A basic and important kind of problems in mathematics that of *classification.* Suppose we would like to formulate a classification problem in classical topology, i.e. in the category Top. A nat-

	Space	Sheaf/algebra	Category	Triangulated envelope
Global	X	$\mathcal{O}_X$	$\mathsf{Vect}(X) \subset \mathsf{QCoh}(X)$	$\mathcal{D}(\mathsf{QCoh}(X)) = \mathcal{D}(\mathsf{Vect}(X))$
Local	$\operatorname{Spec} A$	A	${}_A\mathsf{proj} \subset {}_A\mathsf{Mod}$	$\mathcal{D}({}_A\mathsf{Mod}) := \mathsf{Ch}({}_A\mathsf{Mod})[W^{-1}]$
Example	$\mathbb{P}^n$	$\mathscr{A}$	${}_{\mathscr{A}}\mathsf{Mod} = \mathsf{QCoh}(\mathbb{P}^n)$	$\mathcal{D}(\mathsf{QCoh}(\mathbb{P}^n)) \simeq \mathcal{D}({}_{\mathscr{A}}\mathsf{Mod})$

Table 1: Spaces in various disguises. For the presentation of the projective space $\mathbb{P}^n$ as a non-commutative affine space, the corresponding algebra is the Beilinson quiver algebra $\mathscr{A} = \operatorname{End}\left(\bigoplus_{k=1}^n \mathcal{O}_{\mathbb{P}^n}(k)\right)^{op}$.

ural attempt is to classify all topological spaces up to homeomorphisms, which are precisely the isomorphisms in Top. However...

Remark 3.1. *This problem is unsolvable.*

One way to refine the rough statement in Remark 3.1 is through the notion of algorithmic undecidability: Even for very restricted classes of spaces, such as finite simplicial complexes, there is no algorithm that always decides whether they are homeomorphic (this is a theorem of Markov and others in the 1950s–60s).

We might pose a weaker form of classification problem by e.g. linearizing the category Top. For instance, declare that a continuous map $f : X \to Y$ is an 'isomorphism' if it induces an isomorphism on such algebraic invariants as homotopy, rational homotopy, homology, etc. Specifically, we are imagining a functor

$$\beta : \mathrm{Top} \to \mathsf{C} \tag{32}$$

such that the image $\beta(f)$ of a morphism is an isomorphism in C if and only if $\pi_*(f)$ (or $\pi_*(f) \otimes_{\mathbb{Z}} \mathbb{Q}$, $H_*(f)$, etc.) is an isomorphism in the category Grp of groups. Furthermore, C is required to be initial among such categories. If at least one such category exists, it will be unique due to its universal property. It is denoted by $\mathsf{C} = \mathsf{Top}_*[W^{-1}]$, where W is the set of those morphisms in Top that are inverted by the above criterion.

Example 3.2. *Depending on the setting, one can also consider simplicial sets* $\mathsf{sSet}[W^{-1}]$*, pointed CW complexes* $\mathsf{CW}_*[W^{-1}]$*, etc. instead of* $\mathsf{Top}_*[W^{-1}]$*. If the morphisms in W are weak homotopy equivalences in* CW_**, the resulting category is the localization of* CW_* *at the honest homeomorphisms.*

Now in the category $\mathrm{Top}[W^{-1}]$, two spaces being isomorphic has a more tractable criterion: It suffices to check certain linear algebraic invariants of the spaces involved. However,

Remark 3.3. *It is still impossible to classify all topological spaces up to a reasonable set of 'weak equivalences'.*

Remark 3.4. *It might even be the case that the notion of topological space itself is "incorrect". One line of thoughts in this direction is that of condensed sets due to Clausen and Scholze.*

3.2 (Co)homology theories

Recall 3.5. *A homology theory is a collection of functor $F : \mathsf{Top} \to \mathsf{Ab}$ satisfying a bunch of axioms, such as Mayer–Vietoris sequence (which extracts a long exact sequence*

$$\cdots \longrightarrow H_{n+1}(X) \longrightarrow H_n(A \cap B) \longrightarrow H_n(A) \oplus H_n(B) \longrightarrow H_n(X) \longrightarrow \cdots \tag{33}$$

from a short exact sequence

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathbb{Z}_A \oplus \mathbb{Z}_B \longrightarrow \mathbb{Z}_{A\cap B} \longrightarrow 0 \tag{34}$$

) and the Künneth formula

$$F(X \times Y) \simeq F(X) \otimes F(Y). \tag{35}$$

Remark 3.6. *We don't really need Künneth formula as an axiom after learning the Motiv.*

When stating the Künneth formula, we used the symmetric monoidal structure of the category Top.

Definition 3.7. *A triple $(\mathsf{C}, \otimes, 1_{\mathsf{C}})$ is called a symmetric monoidal category if C is a category with a tensor operation $\otimes : \mathrm{obj}(\mathsf{C}) \times \mathrm{obj}(\mathsf{C}) \to \mathrm{obj}(\mathsf{C})$, a tensor unit $1_{\mathsf{C}} \in \mathrm{obj}(\mathsf{C})$ and isomorphisms $X \otimes 1_{\mathsf{C}} \simeq X \simeq 1_{\mathsf{C}} \otimes X$, a braiding $B_{x,y} : x \otimes y \to y \otimes x$ such that, in addition to various compatibility conditions, the braiding is symmetric:*

$$B_{y,x} \circ B_{x,y} = 1_{x \otimes y}. \tag{36}$$

Example 3.8. Top *has a symmetric monoidal structure* $(\mathsf{Top}, \times, *)$.

Example 3.9. *The category Vec_k of k-vector spaces has a symmetric monoidal structure* $(\mathsf{Vec}_k, \otimes, k)$.

Observation 3.10. *A reduced homology theory is a functor from pointed spaces*

$$F_* : \mathsf{Top}_* \to \mathsf{Ab} \tag{37}$$

Mayer–Vietoris sequence amounts to excision, the Kunneth formula needs to be $F_*(X \wedge Y) = F_*(X) \otimes F_*(Y)$ where $\wedge$ is the smash product.

The relevant symmetric monoidal category for reduced homology theories are that of *pointed* topological spaces $(\mathsf{Top}_*, \wedge, S^0 = * \coprod *)$. Here, the wedge product $\wedge$ is defined by

$$\wedge : \begin{array}{ccc} \mathsf{Top}_* \times \mathsf{Top}_* & \to & \mathsf{Top}_* \\ ((X, x_0), (Y, y_0)) & \mapsto & \dfrac{X \times Y}{X \times \{y_0\} \cup \{x_0\} \times Y} \end{array} \tag{38}$$

The pointed category Top_* is the first stage of a sequence of steps to "linearize" the category Top. In particular, recall that an abelian category, which is a category of "linear" objects and morphisms, has a zero (i.e. initial and terminal) object.

Observation 3.11. *In Top, $*$ is a terminal object. In Top_*, $*$ is a zero object.*

Once fixing a reduced homology theory F_*, we can study the localization $\mathsf{Top}_*[W_{F_*}^{-1}]$ What properties would this category have? For instance, we may ask the representability of the functor F_* in $\mathsf{Top}_*[W_{F_*}^{-1}]$.

Question 3.12. *Do we have* $F_* \simeq \mathrm{Hom}(X, -)$ *for some* X*?*

Example 3.13. $\pi_n(X) = [(S^n, *), (X, x_0)]$ *is already representable in* Top_*.

Suppose we further have Hom-tensor adjunction

$$[X \wedge Y, Z] \simeq [X, [Y, Z]]. \tag{39}$$

In fact, this can be achieved by considering only the compactly generated spaces. In this 'cartesian closed' category, we have

$$\pi_{n+1}(X) = [S^{n+1}, X] = [S^1 \wedge S^n, X] = [S^n, [S^1, X]] = \pi_n(\Omega X) \tag{40}$$

Upshot: If F_* is representable, i.e. $F_\bullet(X) = [\tilde{F}_\bullet, X]$ $\forall X$ for some $\tilde{F}_n \in \mathsf{Top}_*$, then there is a weak equivalence

$$\Sigma \tilde{F}_n \xrightarrow[\text{(W)}]{\sim} \tilde{F}_{n+1}. \tag{41}$$

This motivates the foollowing definition.

Definition 3.14. *A* <u>*spectrum*</u> X *is a sequence of pointed topological spaces* $\{X_n : n \in \mathbb{N}\}$ *with a collection of homotopy equivalences*

$$f_n : \Sigma X_n \xrightarrow{\sim} X_{n+1}. \tag{42}$$

Theorem 3.15.

- *There exists a well-defined closed (i.e. having internal Hom and Hom-tensor adjunction) symmetric monoidal category*

$$(\mathsf{Spectra}, \wedge, \mathbb{S}), \quad \mathbb{S} := \{\mathbb{S}^n : n \in \mathbb{N}\} \tag{43}$$

  which is an enhanced version of $(\mathsf{Top}_*, \wedge, S^0)$.

- *For every homology theory* $F_* : \mathsf{Top}_* \to \mathsf{Ab}$*, there exists a unique spectrum* $\tilde{F}_\bullet$ *such that* $F_\bullet(X) = [\tilde{F}_\bullet, X]$ *That is, the category of spectra is the category of homology theories over* Top_*.

Taking $(\mathsf{Spectra}, \wedge, \mathbb{S})$ as the domain of (co)homology theories, we move one more step forward to linearize the category Top: With the shift operations $X_\bullet \to X_{\bullet\pm1}$, it now behaves more like the category $\mathsf{Ch}_\mathbb{Z}$ of chain complexes of abelian groups. Furthermore, degree shifts of spectra respect the given homology theory F_*.

There is one more step to go, as suggested by the following observation.

Observation 3.16. *There is a pair of adjoint functors* $(\Sigma \vdash \Omega)$ *that induce the shifts of degree in homology and homotopy:*

$$H_\bullet(\Sigma X) = H_{\bullet-1}(X), \tag{44a}$$

$$\pi_{\bullet+1}(X) = \pi_\bullet(\Omega X). \tag{44b}$$

The shift operations are obviously invertible. Thus, we would have $\Sigma = \Omega^{-1}$ *in the localized category* $\mathsf{Top}_*[W_{F_*}^{-1}]$.

3.3 Stabilization

Let C be a category with a terminal object $* \in \mathsf{C}$. Assume further that C is complete (i.e. has limits). In particular, C is equipped with fiber product, and one can define $X \times Y := X \times_* Y$. For example, we consider $(\mathsf{Top}, *)$

Definition 3.17. *Given a category C with a terminal object $* \in \mathsf{C}$, the comma category C_* over $*$ is defined by*

- $\mathrm{obj}(\mathsf{C}_*) = \{* \to X\}$
- $\mathrm{mor}(* \to X, * \to Y) = \left\{ \begin{tikzcd} * \arrow[r] \arrow[dr] & X \arrow[d] \\ & Y \end{tikzcd} \right\}$

Example 3.18. Top_* *is a comma category obtained from* Top*. Moreover, the comma construction naturally led us to the symmetric monoidal category* $(\mathsf{Top}_*, \wedge, S^0)$.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X \arrow[ul, phantom, "\ulcorner"] \end{tikzcd} \qquad \begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & * \arrow[d] \\ * \arrow[r] & X \end{tikzcd} \tag{45}$$

If computed in the homotopy category Top_*, ΣX and ΩX turn out to be the usual suspension and looping.

What we want to make true is $[\Sigma = \Omega^{-1}]$ in the category $\mathsf{Top}_*[W_{F_*}^{-1}]$. Why? Because Σ and Ω are just shifts in the opposite direction on the level of homology. Considering weak equivalences induced by the homology theory, we want to make $[\Sigma = \Omega^{-1}]$ on the level of spaces.

Proposition 3.19. *In the category $\mathsf{C}[W_F^{-1}]$ obtained by localizing C at the weak equivalences coming from the cohomology theory F, the based loop Ω and reduced suspension Σ functor are self equivalences of $\mathsf{C}[W_F^{-1}]$ which are quasi-inverse to each other.*

Proof. The suspension functor induces a shift of degrees in cohomology, which has a simple inverse (shift in the reverse direction). Hence, suspension becomes an equivalence in the localized category. Its inverse must coincide with the right adjoint functor Ω. Specifically, say $(F \vdash G)$ is an adjoint pair and F and an inverse F^{-1}. Then we can construct a natural isomorphism $F^{-1} \Rightarrow G$ by using the correspondence

$$\mathrm{Hom}(x, F^{-1}y) \simeq \mathrm{Hom}(Fx, y) \simeq \mathrm{Hom}(x, Gy). \tag{46}$$

□

$\Sigma = \Omega^{-1}$ implies in particular that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & Y \end{tikzcd} \tag{47}$$

is pullback if and only if pushout.

Example 3.20. *Let* $A, B, C \in {}_R\mathsf{Mod}$ *where* $R \in \mathsf{CRing}$. *Consider the diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "g"] \\ 0 \arrow[r] & C \end{tikzcd} \tag{48}$$

Then

- *the diagram commutes* $\iff$ $A \xrightarrow{f} B \xrightarrow{g} C$ *is a chain complex.*
- *it is a pullback diagram* $\iff$ $A = \ker(g)$.
- *it is a pushout diagram* $\iff$ $C = \operatorname{coker}(f)$.

In particular, the diagram is pullback and pushout if and only if $A \xrightarrow{f} B \xrightarrow{g} C$ *is a short exact sequence.*

Definition 3.21. *A category* $(\mathsf{C}_*, \cdots)$ *is* <u>*stable*</u> *if and only if pushout coincides with pullback.*

- *General* C_* *is not stable, e.g.* $(\mathsf{Top}_*, \wedge, S^0)$
- *However, one can "stabilize" such a category in a universal way.*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] & & \\ * \arrow[r] & \Sigma X \arrow[r] \arrow[d] & * \arrow[d] & \\ & * \arrow[r] & \Sigma^2 X \arrow[r] \arrow[d] & * \\ & & * & \ddots \end{tikzcd} \tag{49}$$

We take the whole diagram as an object of $\mathsf{Stab}(\mathsf{C}_*)$.

Example 3.22. *If* $\mathsf{C}_* = {}_R\mathsf{Mod}$, *then* $\mathsf{Stab}({}_R\mathsf{Mod}) = \mathsf{Ch}({}_R\mathsf{Mod})$. *In this case, the suspension functor is the shift map*

$$\Sigma : (M^\bullet, d) \mapsto (M^{\bullet+1}, -d). \tag{50}$$

4 Geometry of categories

4.1 First contact with spectral algebraic geometry

At the beginning there were closed symmetric monoidal categories...

At the starting point of our story is the question, "What is $\mathcal{D}(\mathsf{QCoh}(X))$?" Could we begin with categories in the first place to study geometry, without referring to sets?

Recall 4.1. *By calling* C *a category, we mean that there is the following data.*

- $\mathrm{obj}(\mathsf{C}) \ni X, Y$

- $\mathrm{mor}(X, Y)$
- $1_X \in \mathrm{mor}(X, X)$
- $\circ : \mathrm{mor}(Y, Z) \times \mathrm{mor}(X, Y) \to \mathrm{mor}(X, Z)$

But it seems that we need auxiliary data to clarify what these mean. So, What were in the beginning are not categories, but closed symmetric monoidal categories.

Definition 4.2 (refined version)**.** *C is a category when there are structures*

- $\mathrm{mor}(X, Y)$*, an object of the closed symmetric monoidal category* $(\mathsf{Set}, \times, *)$
- $1_X : * \to \mathrm{mor}(X, X)$*, a morphism in* $(\mathsf{Set}, \times, *)$
- $\circ : \mathrm{mor}(Y, Z) \times \mathrm{mor}(X, Y) \to \mathrm{mor}(X, Z)$*, a morphism in* $(\mathsf{Set}, \times, *)$

In this viewpoint, closed symmetric monoidal category is a single primitive concept needed for our purpose—even preceding the concept of sets.

Now, Def. 4.2 is succinctly generalized into enriched categories.

Definition 4.3. *Let* $(\mathsf{E}, \otimes_{\mathsf{E}}, 1_{\mathsf{E}})$ *be a closed symmetric monoidal category. A category* C *enriched over* E *consists of the data*

- $\mathrm{mor}(X, Y) \in \mathrm{obj}(\mathsf{E})$
- $\mathrm{mor}(Y, Z) \otimes_{\mathsf{E}} \mathrm{mor}(X, Y) \to \mathrm{mor}(X, Z)$
- $1_X : 1_{\mathsf{E}} \to \mathrm{mor}(X, X)$

such that certain diagrams in E *commute (associativity, etc.)*

We will denote by $\mathsf{Cat}(\mathsf{E}) :=$ the category of categories enriched over E. If for example $A \in {}_k\mathsf{Alg}$, then ${}_A\mathsf{Mod}$ is enriched over $({}_k\mathsf{Mod}, \otimes_k, k)$.

Suppose $f : (\mathsf{E}, \otimes_{\mathsf{E}}, 1_{\mathsf{E}}) \to (\mathsf{F}, \otimes_{\mathsf{F}}, 1_{\mathsf{F}})$ is a monoidal functor. Then we can transfer categories from $\mathsf{Cat}(\mathsf{E})$ to $\mathsf{Cat}(\mathsf{F})$,

$$\tilde{f} : \begin{array}{ccc} \mathsf{Cat}(\mathsf{E}) & \to & \mathsf{Cat}(\mathsf{F}) \\ \mathsf{C} & \mapsto & \tilde{\mathsf{C}} \end{array} \tag{51}$$

by setting $\mathrm{obj}(\tilde{\mathsf{C}}) = \mathrm{obj}(\mathsf{C})$ and $\mathrm{mor}_{\tilde{\mathsf{C}}}(X, Y) = f(\mathrm{mor}_{\mathsf{C}}(X, Y))$. Tensor unit and composition are well-defined due to the properties of monoidal functor.

Example 4.4. *A homology theory* $F_* : (\mathsf{Top}_*, \wedge, S^0) \to (\mathsf{Ab}, \otimes, \mathbb{Z})$ *is monoidal by the Künneth formula. This induces a functor*

$$\tilde{F}_* : \mathsf{Cat}(\mathsf{Top}_*) \to \mathsf{Cat}(\mathsf{Ab}) \tag{52}$$

In particular, we can take $F_* = \pi_*$ in Example 4.4. Upon stabilization, which we described in Sec. 3.3, we get an enhanced functor

$$\tilde{\pi}_* : \mathsf{Cat}(\mathsf{Spectra}) \to \mathsf{Cat}(\mathsf{Ch}(\mathsf{Ab})) \tag{53}$$

In fact:

$$\mathcal{D}(\mathsf{QCoh}(X)) \in \tilde{\pi}_* \mathsf{Cat}(\mathsf{Spectra}) \tag{54}$$

That is, the morphism sets of $\mathcal{D}(\mathsf{QCoh}(X))$ are $\pi_*(M)$ for some spectrum M. Since we are considering spectra M, no information is lost upon passing to the homotopy groups $\tilde{\pi}_*[M]$.

4.2 Further generalization

Geometry	Space	Algebra	Category	Stabilization
Commutative	X	Sheaf $\mathcal{O}_X$	$\mathsf{Vect}(X) \subset \mathsf{QCoh}(X)$	$\mathcal{D}(\mathsf{QCoh}(X))$
Noncommutative	Spec $\mathscr{A}$	category $\mathscr{A}$	${}_A\mathsf{proj} \subset {}_A\mathsf{Mod}$	$\mathcal{D}({}_{\mathscr{A}}\mathsf{Mod})$

Table 2: More compact version of Table. 1.

Recall that the conventional commutative geometry is locally given by a commutative algebra A. For the non-commutative framework to subsume the conventional geometry, there must be a procedure that changes a commutative algebra A into a category, which is a building block of non-commutative spaces.

Observation 4.5. *An algebra is a category with one object: Given $A \in \mathsf{Alg}$, define $B(A^{op}) \in \mathsf{Cat}$ by*

- $\mathrm{obj}(B(A^{op})) = \{*\}$
- $\mathrm{mor}(*, *) = A$
- $\circ = \mathrm{mult} : A \otimes A \to A$

Conversely, a (small) category is an "algebra with many objects".

- $\mathscr{A}$: a "A_∞-category" or dg-category.
- $\mathsf{QCoh}(\mathrm{Spec}(A)) := {}_A\mathsf{Mod} = \mathsf{Fun}(B(A^{op}), {}_k\mathsf{Mod})$ when $A \in {}_k\mathsf{Alg}$.
- $\mathsf{QCoh}(\mathrm{Spec}(\mathscr{A})) := {}_{\mathscr{A}}\mathsf{Mod} := \mathsf{Fun}(\mathscr{A}^{op}, {}_k\mathsf{Mod})$

Definition 4.6 (dualizable object). $(\mathsf{E}, \otimes_\mathsf{E}, 1_\mathsf{E}) \ni X$ *is <u>dualizable</u> if and only if* $\exists X^\vee \in \mathsf{E}$ *and* $\exists\, \mathrm{ev} : X \otimes_\mathsf{E} X^\vee \to 1_\mathsf{E}$, ...

Example 4.7.

- $({}_k\mathsf{Mod}, \otimes_k, k) \ni V$ *is dualizable if and only if V is finite dimensional. In this case,* $V^\vee = \mathrm{Hom}(V, k)$.
- $(\mathsf{dgCat}_k, \boxtimes_k, \mathsf{Ch}({}_k\mathsf{Mod})) \ni \mathscr{A}$ *is dualizable, and* $\mathscr{A}^\vee = \mathscr{A}^{op}$.

One can take the dual once more in a bigger category $\mathsf{Cat}(\mathsf{Ch}({}_k\mathsf{Mod}))$.

$$\mathsf{Cat}(\mathsf{Ch}({}_k\mathsf{Mod})) \ni \mathscr{A}^{\vee\vee} = \mathrm{Hom}(\mathscr{A}^\vee, {}_k\mathsf{Mod}) \tag{55}$$

Remark 4.8. *In fact, $\mathscr{A}$ is not dualizable on the nose in the bigger category. As an illustration, consider an infinite-dimensional vector space $V \hookrightarrow V^{\vee\vee}$. Although V itself may not be reflexive under the dualizing operation $V \mapsto V^\vee$, the closure of V inside $V^{\vee\vee}$ would be. In analogy, we take the closure of $\mathscr{A} \hookrightarrow \mathscr{A}^{\vee\vee}$, where $\hookrightarrow$ is the Yoneda embedding $\mathcal{Y}$. The image $\mathcal{Y}(A)$ is a free module over A, and the closure would be a "projective" module.*

Given ${}_A\mathsf{Mod} = \mathrm{Fun}(B(A^{op}), {}_k\mathsf{Mod})$, then we naturally consider the representable functors (a la Yoneda). But $B(A^{op})$ has only a single object, so

$$F(*) = \mathrm{Hom}(*, *) = A \tag{56}$$

produces a free A-module. Free modules with higher rank $n > 1$ are allowed if $B(A^{op})$ is extended with direct sums of objects. Upshot: Free module = representable functor.

Hence, we define the would-be category of free modules over $\mathscr{A}$ by

$${}_{\mathscr{A}}\mathsf{free} := \mathsf{Triv}(\mathrm{Spec}(\mathscr{A})) = \text{ image of Yoneda functor } \mathcal{Y}(\mathscr{A}) \subset {}_{\mathscr{A}}\mathsf{Mod}. \tag{57}$$

In fact, one can observe that $\mathsf{Triv}(\mathrm{Spec}(\mathscr{A}))$ is not complete in $\mathscr{A}^{\vee\vee}$. Further motivated by the fact that projective modules are precisely the direct summands of free modules, i.e. colimits of free modules, we define ${}_{\mathscr{A}}\mathsf{proj}$ by

$$\text{“}{}_{\mathscr{A}}\mathsf{proj}\text{”} := \mathsf{Perf}(\mathscr{A}) := \text{completion of } \mathsf{Triv}(\mathrm{Spec}(\mathscr{A})) \text{ in } \mathscr{A}^{\vee\vee}. \tag{58}$$

One thus talks about “perfect complexes” or “perfect objects”.

Slogan 4.9. “${}_{\mathscr{A}}\mathsf{proj}$” = *completion of* ${}_{\mathscr{A}}\mathsf{free}$ *in* ${}_{\mathscr{A}}\mathsf{Mod}$.

Finally, we enhance the category ${}_k\mathsf{Mod}$ to $\mathsf{Ch}({}_k\mathsf{Mod})$ in the above developments.

Remark 4.10. *A dg-category $\mathscr{A}$ is not triangulated, hence not a stable category in general. But upon completion (closure inside $\mathscr{A}^{\vee\vee}$), the category in question becomes enriched over* $\mathsf{Ch}({}_k\mathsf{Mod})$ *which is stable. So these ideas surprisingly converge with the stabilization of categories.*

Note 4.11. *Every A_∞-category is quasi-isomorphic to a dg-category through a procedure known as homotopy transfer. Conversely, a dg-category is an A_∞-category. A dg-category $\mathscr{A}$ is not stable in general, but the module category of $\mathscr{A}$ does. Surprisingly, under suitable conditions, this module category coincides with the stabilization of $\mathscr{A}$.*

The collection of categories of the form $\mathscr{A}^{\vee\vee}$ has its own name, which is the category of left presentable stable k-linear categories.

$$\{\mathsf{Perf}(\mathscr{A})\} \subset \mathsf{Pr}_{L,k} = \{\mathscr{A}^{\vee\vee}\} \subset \mathsf{Cat}(\mathsf{Ch}({}_k\mathsf{Mod})) \tag{59}$$

Question 4.12. $\mathsf{Perf}(\mathscr{A}) = \mathsf{Pr}_{L,k}$*? If not, what is the precise relation?*

Answer: In fact, dualizable objects of $\mathsf{Pr}_{L,k}$ are the “localizations” of $\mathsf{Perf}(\mathscr{A})$. In this sense, $\mathsf{Pr}_{L,k}$ is the setting for all geometries we study.

4.3 A flyby of motives

In general, we define a “sheaf theory” by specifying its 3-functor or 6-functor formalism. A sheaf theory is a functor from E_∞-geometry to E_1-geometry. E_1-spaces are categories by definition, so the target of a sheaf theory is a category of categories. The domain is the collection of E_∞-spaces, which turns out to be the category of correspondences. Since sheaf theories are different ways of linearization, all sheaf theories should factor through a universal linearization of the category of correspondences.

Definition 4.13. *A sheaf theory is a monoidal functor*

$$\mathcal{S} : \mathsf{Corr}(k) \to \mathsf{Cat}(\mathsf{Ch}({}_k\mathsf{Mod})). \tag{60}$$

There is a universal linearization of the category $\mathsf{Corr}(k)$*, called the category of motives* $\mathsf{Motiv}(k)$*, through which all sheaf theories on* $\mathsf{Corr}(k)$ *factor.*

$$\begin{tikzcd} \mathsf{Corr}(k) \arrow[rr, "\mathcal{S}"] \arrow[dr] & & \mathsf{Cat}(\mathsf{Ch}({}_k\mathsf{Mod})) \\ & \mathsf{Motiv}(k) \arrow[ur, dotted] & \end{tikzcd} \tag{61}$$

Choosing a sheaf theory is equivalent to specifying a 6-functor formalism. In string theory jargon, this amounts to fixing a D-brane, i.e. an object of $\mathsf{QCoh}(X)$.

Example 4.14. *The following are sheaf theories.*

- $X \mapsto \mathsf{Loc}(X)$*, the local systems on* X
- *The type IIA string theory assigns* $X \mapsto \mathsf{Fuk}(X)$*, the Fukaya category of* X*.*
- *The type IIB string theory assigns* $X \mapsto \mathsf{QCoh}(X)$*, the coherent sheaves on* X*.*

4.4 Epilogue

- Another name of A_∞-algebra is the E_1-algebra. As we move from the commutative geometry to non-commutative geometry, we replace the charts being E_∞-algebras by E_1-algebras. This is equivalent to discarding the commutativity structures in the initial E_∞-algebra. An E_1-algebra is in turn equivalent to a dg-category, so we use dg categories as local chart of a geometric object. At the same time, morphisms between such objects are enhanced to functors between dg-categories.
- As a corollary, we may take non-commutative algebras such as operator algebras to construct a geometric object, which leads to a geometric treatment of quantum mechanics (1D QFT).
- Yet another perspective is that one can construct E_∞ algebra out of a E_1 algebra, which is called the *moduli of objects*. In the large-N limit, certain information is preserved, which leads to the gauge theory – open string duality. This is an example of holography.