Every L has their own story 1 : L-function connecting Number Fields and Elliptic Curves

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Outline

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Introduction to the Seminar

- L-functions : 깊고 심오한... 수학의 아름다운 심장

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(다음 번엔 아마도 Hasse-Weil L-function?)

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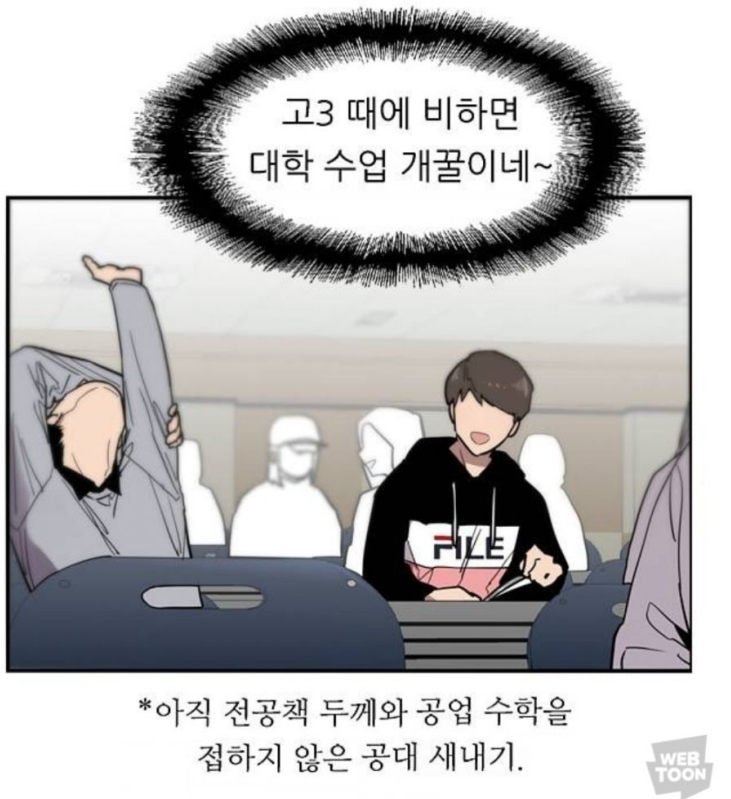
- 아직 귀여운 신입생 ⇒ 저도 모로는 게 많아요...



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Hence, we have to know what $\mathbb{Z}[i]$ "looks" like. Similarly, for other number theoretic questions, knowing structures like $\mathbb{Z}, \mathbb{Z}[i]$ helps.

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Theorem(Chebotarev density theorem).

For number field K with degree n, ratio of $\mathbb{Z}$-primes that remain prime is $\frac{1}{n}$.

Remark

$\mathbb{Z}[i]$: From FTST, every prime $p \equiv 1 (\mathrm{mod}\, 4)$ is $a^2 + b^2 = (a - bi)(a + bi)$, hence not prime. As mentioned, $p \equiv 3$ remains prime.
$(2 = -i(1 + i)^2$ is ramified)
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Definition(Discriminant of Number field).

$$D_K = \det \begin{pmatrix} \sigma_1(b_1) & \cdots & \sigma_1(b_n) \\ \vdots & \ddots & \vdots \\ \sigma_n(b_1) & \cdots & \sigma_n(b_n) \end{pmatrix}^2 \in \mathbb{Z}$$

where b_i are integral basis of $\mathcal{O}_K$, $\sigma_i : K \to \mathbb{C}$ are injective ring homomorphisms.

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$\Rightarrow$2 is ramified in $\mathbb{Q}(i)$ since $2 | \det \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}^2 = -4$

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Intuition : F_K, P_K is group of ideals/principal ideals.
(For those who wonders, $I \subset R$ for ring R is fractional ideal if aI is ideal of R for some $a \in R$. If that ideal is principal, it's principal fractional ideal.)

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Remark 2. is from the fact that every $\mathcal{O}_K$ is Dedekind domain.

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Hence, we can answer

- Is $\mathcal{O}_K$ UFD?

by calculating h_K... which is difficult.

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For number field K, we define **Dedekind zeta function** $\zeta_K(s)$ as

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$\zeta_K(s)$ shares good conditions with $\zeta(s)$.

Properties of $\zeta(s)$

- Dirichlet series converges on $\mathrm{Re}(s) > 1$
- Reflection formula :
  $\xi(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s) = \xi(1-s)$
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- Reflection formula : $\Lambda_K(s) = |D_K|^{\frac{s}{2}}\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta_K(s) = \Lambda_K(1-s)$
- Hence, analytically continued to $\mathbb{C}$, except for the pole $s = 1$.

Where $\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})$, $\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s)$, r_1, r_2 is the number of real/complex(up to conjugation) homomorphism of K.

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$$\operatorname*{Res}_{s=1} \zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} \cdot Reg_K \cdot h_K}{\sqrt{|D_K|} \cdot w_K}$$

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e.g. $K = \mathbb{Q}(i)$. Well known : $Res_{s=1}\zeta_{\mathbb{Q}(i)} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$. So,

$$\begin{aligned} \frac{\pi}{4} &= \frac{2^0 \cdot (2\pi)^1 \cdot 1 \cdot h_{\mathbb{Q}(i)}}{\sqrt{|-4|} \cdot 4} \\ &= \frac{\pi}{4} \cdot h_{\mathbb{Q}(i)}, \end{aligned}$$

hence $h_{\mathbb{Q}(i)} = 1$, i.e. $\mathbb{Z}[i]$ is UFD.(Overkill...)

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Solution set of $E : y^2 = x^3 + ax + b$ for $a, b \in \mathbb{Q}$ with $4a^3 + 27b^2 \neq 0$ is called the **Elliptic Curve**.($E(\mathbb{C})$, $E(\mathbb{Q})$,etc.)

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Q. Why EC is important?

Elliptic Curves

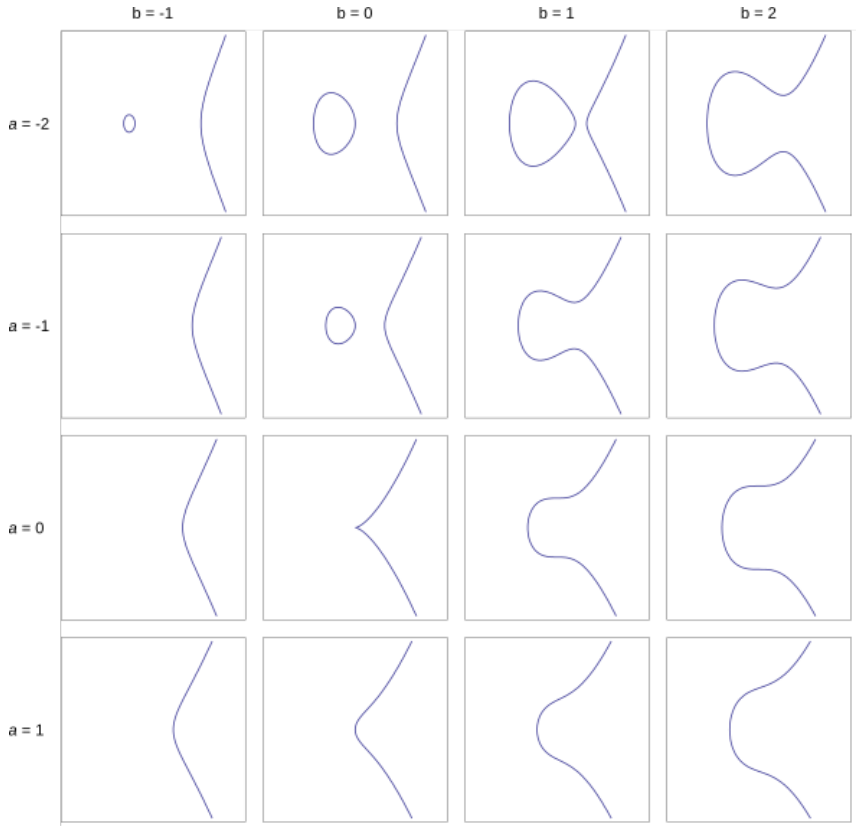
Definition

Solution set of $E : y^2 = x^3 + ax + b$ for $a, b \in \mathbb{Q}$ with $4a^3 + 27b^2 \neq 0$ is called the **Elliptic Curve**.($E(\mathbb{C})$, $E(\mathbb{Q})$,etc.)

Q. Why EC is important?

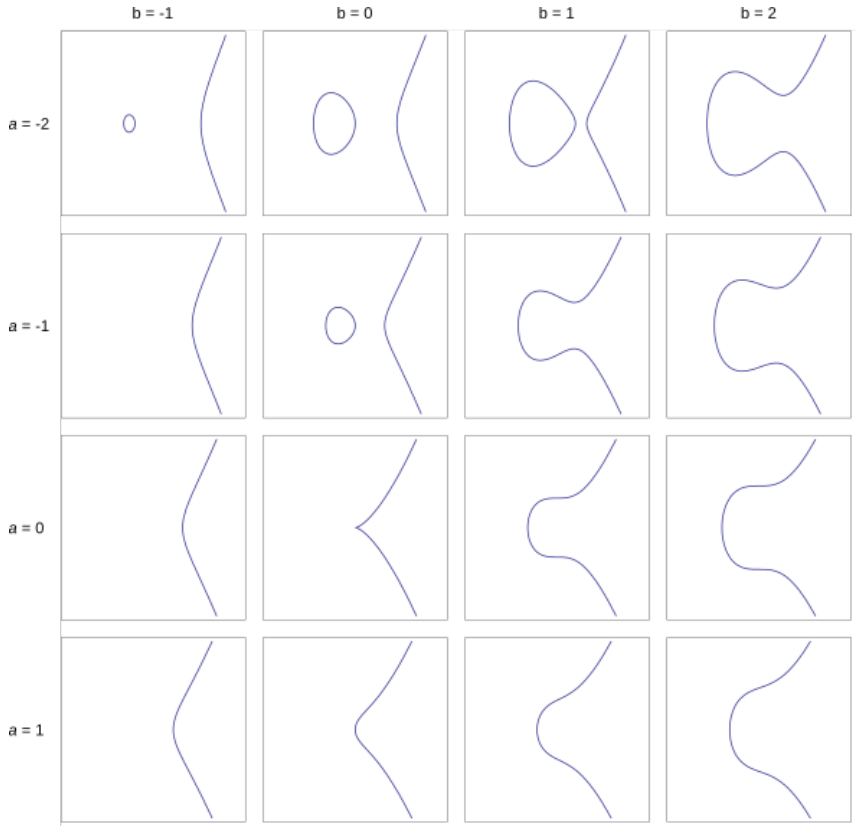
A. Unique unsolved and unknown Diophantine eq. with 2 variables.

Elliptic Curves



In terms of # of $\mathbb{Q}$-Solution set :

Elliptic Curves



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- degree 1,2 - ∞ or 0(trivial and easy)

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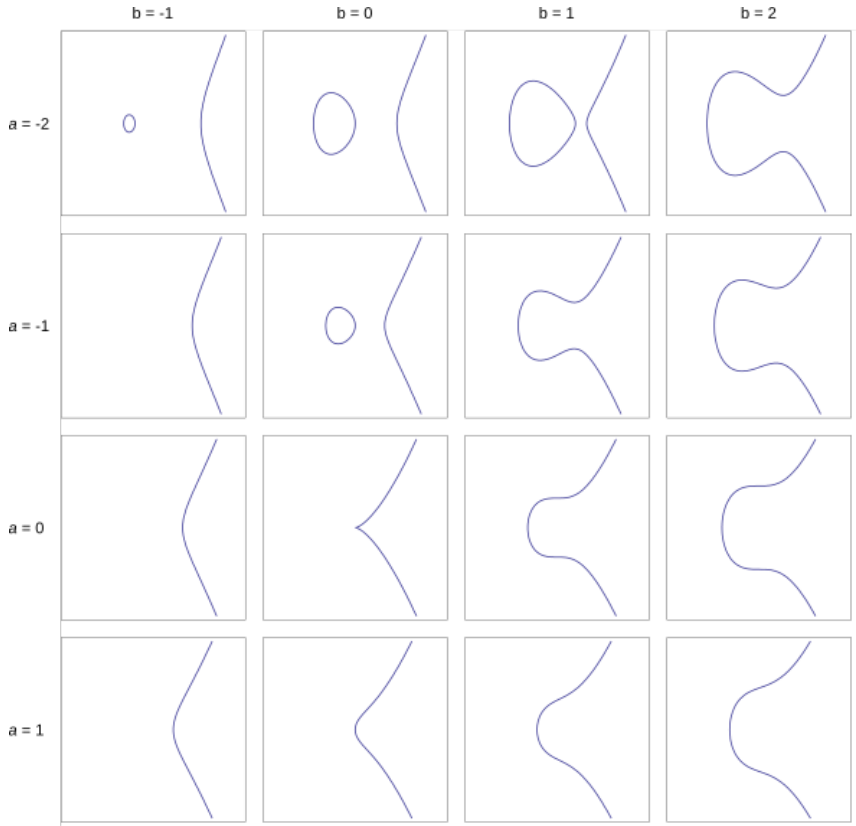

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- degree $\geq$ 4 - finite!(Faltings, 1983)

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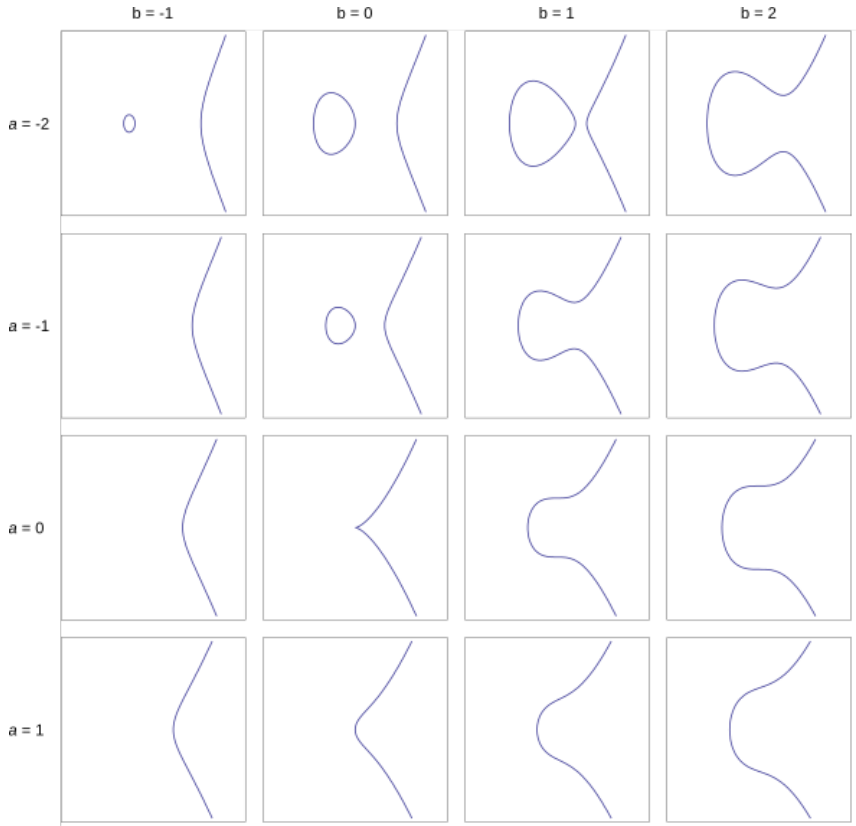

In terms of # of $\mathbb{Q}$-Solution set :

- degree 1,2 - ∞ or 0(trivial and easy)
- degree $\geq$ 4 - finite!(Faltings, 1983)
- degree 3 - 0 or finite or ∞ = 엄..

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Our interest $= E(\mathbb{Q}) = \{(x, y) \in \mathbb{Q}^2 : y^2 = x^3 + ax + b\} \cup \{\infty\}$

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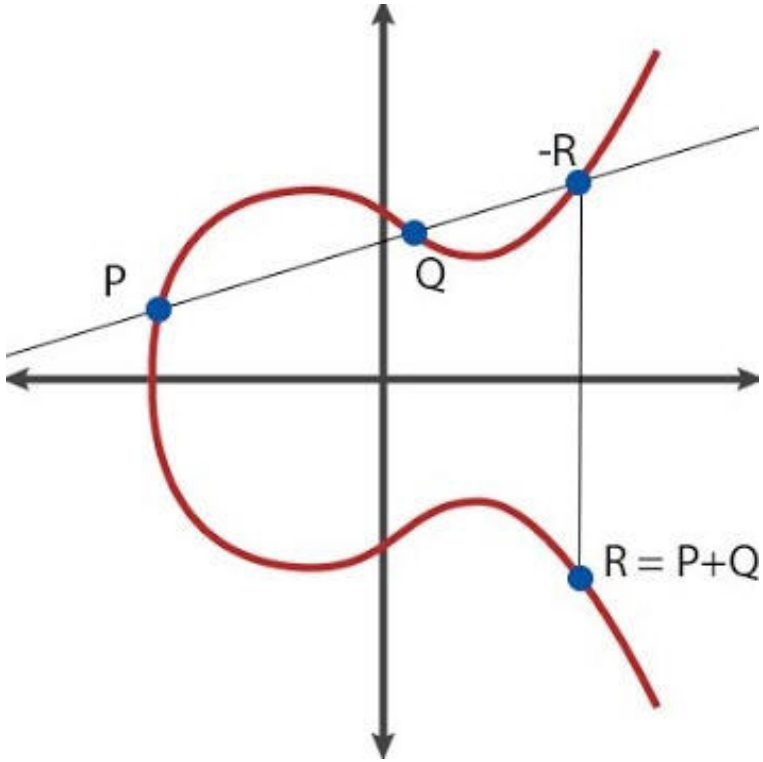

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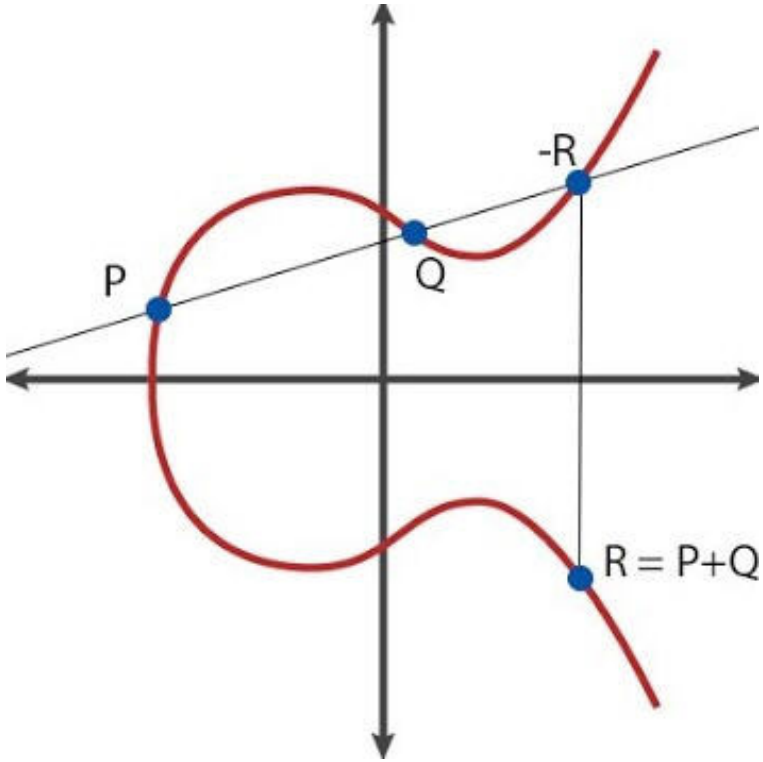

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- We call them **Mordell-Weil group** of E.

Elliptic Curves

Theorem(Mordell-Weil Theorem).

For every elliptic curve E, $E(\mathbb{Q})$ is finitely generated abelian group. i.e., we can write $E(\mathbb{Q}) \cong \mathbb{Z}^r \oplus E(\mathbb{Q})_{\text{tor}}$

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Theorem(Mazur's torsion theorem).

$E(\mathbb{Q})_{\text{tor}}$ is isomorphic to one of these :

- $\mathbb{Z}/n\mathbb{Z}$ for $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12$
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Theorem(Elkies & Zev Klagsbrun ; 2024).

The rank of the elliptic curve $E_{29} : y^2 + xy = x^3 - ax + b$ is ≥ 29, and equality holds if Riemann hypothesis for $L(E, s)$ holds true.

Elliptic Curves

Here, the L-function for elliptic curve is defined as follows.

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Definition(Hasse-Weil L-function for Elliptic Curve).

For elliptic curve E, **Hasse-Weil L-function** of $L(E,s)$ is defined as the following Euler product

$$L(E,s) = \prod_p L_p(E,s)^{-1}$$

where

$$L_p(E,s) = \begin{cases} 1 & \text{if } p^2 \mid N \\ 1 \pm p^{-s} & \text{if } p \mid\mid N \\ 1 - a_p p^{-s} + p^{1-2s} & \text{if } p \nmid N \end{cases}$$

and $a_p := (p+1) - \#E(\mathbb{F}_p)$, N is the conductor of E.

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c.f. E remains elliptic curve(i.e., has no multiple zero) on $\mathbb{F}_p$ iff $p \nmid N$

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- Dirichlet series converges on $\mathrm{Re}(s) > 1$
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  $\xi(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s) = \xi(1-s)$
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Properties of $L(E, s)$

- Dirichlet series converges on $\mathrm{Re}(s) > \frac{3}{2}$
- Reflection formula : $\Lambda(E, s) = N^{\frac{s}{2}}(2\pi)^{-s}\Gamma(s)L(E, s) = \omega\Lambda(E, 2-s)$ for some $\omega = \pm 1$. (This is from Modularity theorem!)
- Hence, analytically continued to $\mathbb{C}$ except for some poles.

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- BSD connects the algebraic data(Mordell-Weil group) with analytic data(Hasse-Weil L-function).

Elliptic Curves

One way to calculate rank$E(\mathbb{Q})$: descent!(for short, calculating $E(\mathbb{Q})/mE(\mathbb{Q})$). In the process, we need to know one more invariant about EC.

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We define Tate-Shafarevich group of an elliptic curve E as following :

$$Ш(E/\mathbb{Q}) := \bigcap_{\text{place } v} \ker\left(H^1(G(\mathbb{Q}), E) \to H^1(G(\mathbb{Q}_v), E_v)\right)$$

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Precise definition of $Ш$... skip.

- Intuition : $Ш$ measures the failure of local-global principal!

Outline

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Object in EC and NF **aligns**.

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Number Field : Theorem(Dirichlet unit theorem).

$\mathcal{O}_K^\times$ is finitely generated. i.e., $\mathcal{O}_K^\times \cong \mathbb{Z}^r \oplus (\mathcal{O}_K^\times)_{\mathrm{tor}}$ where $r = r_1 + r_2 - 1$, $(\mathcal{O}_K^\times)_{\mathrm{tor}} = \mathcal{O}_K \cap \mathbb{S}^1$.

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In NF side, we have Reflection formula

- $|D_K|^{\frac{s}{2}}\Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta_K(s) = |D_K|^{\frac{1-s}{2}}\Gamma_{\mathbb{R}}(1-s)^{r_1}\Gamma_{\mathbb{C}}(1-s)^{r_2}\zeta_K(1-s)$
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well well well.... not end yet.

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For every elliptic curve E,

$$\text{rank}E(\mathbb{Q}) = \operatorname*{ord}_{s=1} L(E,s) = r$$

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for Reg_E is the regulator of E, Ω_E is the real period of E, c_p is the Tamagawa number of E at p.

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Now, let's calculate coeff. of ζ_K at $s = 0$

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Hence we get the real correspondence between EC and NF.

The Analogy

Elliptic Curve : Conjecture(Precise BSD).

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- Where are $\Omega_E, \prod_{p|N} c_p$ gone?
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The analogy is clear, but ambiguous.

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그러게요... As mentioned, we'll not talk about how those come from. Just "throw" the topics.

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Big Picture : Scheme

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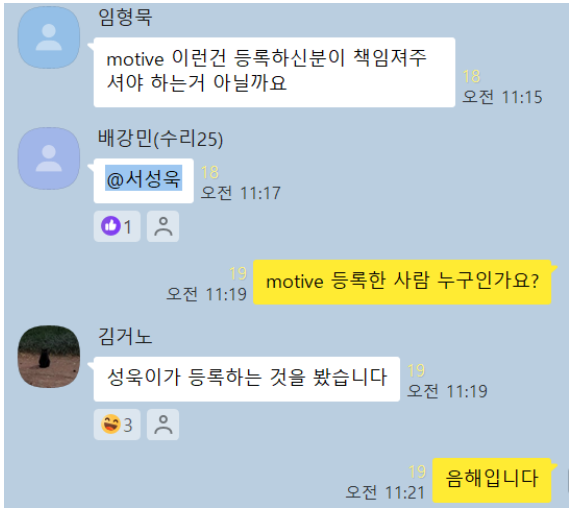
Much Bigger Picture : Motive!

Overview of the analogy



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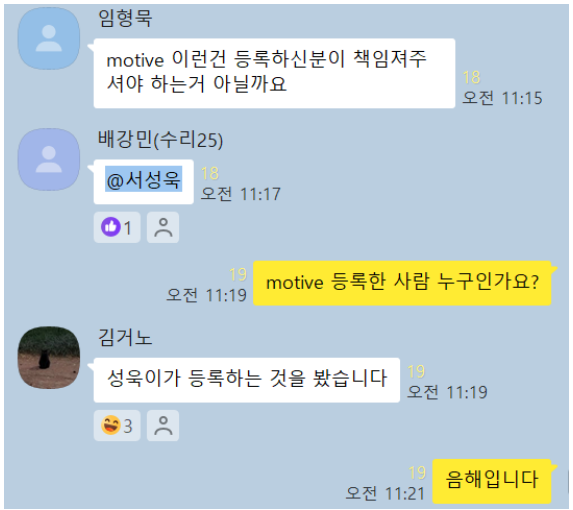

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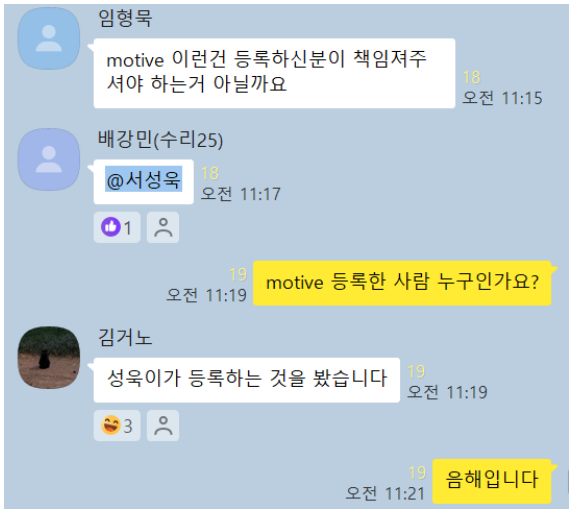

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In the formulation of Grothendieck for smooth projective varieties, a motive is a triple (X, p, m), where X is a smooth projective variety, $p : X \vdash X$ is an idempotent correspondence, and m an integer; however, such a triple contains almost no information outside the context of Grothendieck's category of pure motives, where a morphism from (X, p, m) to (Y, q, n) is given by a correspondence of degree $n - m$. A more object-focused approach is taken by Pierre Deligne in *Le Groupe Fondamental de la Droite Projective Moins Trois Points*. In that article, a motive is a "system of realisations" – that is, a tuple

$$\left(M_B, M_{\mathrm{DR}}, M_{\mathbb{A}^f}, M_{\mathrm{cris},p}, \mathrm{comp}_{\mathrm{DR},B}, \mathrm{comp}_{\mathbb{A}^f,B}, \mathrm{comp}_{\mathrm{cris}\,p,\mathrm{DR}}, W, F_\infty, F, \phi, \phi_p\right)$$

consisting of modules

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over the rings

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respectively, various comparison isomorphisms

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between the obvious base changes of these modules, filtrations W, F, a action ϕ of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}, \mathbb{Q})$ on $M_{\mathbb{A}^f}$, and a "Frobenius" automorphism ϕ_p of $M_{\mathrm{cris},p}$. This data is modeled on the cohomologies of a smooth projective $\mathbb{Q}$-variety and the structures and compatibilities they admit, and gives an idea about what kind of information is contained in a motive.

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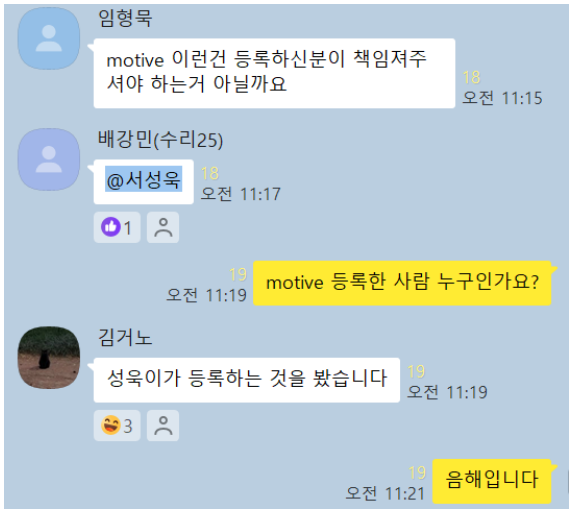

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Conjecture(Bloch-Kato conjecture for L-functions).

$L(s, M)$ has zero at $s = 0$ with order

$$\operatorname*{ord}_{s=0} L(s, M) = \dim H_1^f(G_K, V^*(1)) - \dim H_0(G_K, V^*(1))$$

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- Motivic L-function is one part of **Langlands Program**!
  - More precise : All motivic L is automorphic L.(Like Modularity theorem.)

The End

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