Introduction to BV Formalism
(based on a talk by Myung Jin Jeong)

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March 14, 2025

Keywords: Batalin–Vilkovisky formalism · classical master equation · quantum master equation · gauge theory · graded manifold · shifted cotangent bundle · shifted symplectic structure · Cartan calculus · Berezin integration · derived critical locus · Gerstenhaber algebra

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1 Introduction

The Batalin–Vilkovisky (BV) formalism is a powerful mathematical framework developed to define equations of motion in classical field theories and path integrals in quantum field theories.

Instead of establishing the functional calculus on the plethora of wildly behaving field spaces in physics, BV formalism takes an algebro-geometric route by systematically incorporating both fields and their “antifields” into a graded-geometric structure.

Conventions

Algebras Abstract algebra jargon will occasionally be used in this note. Whenever a ring or and algebra is referred to, it is commutative and equipped with unity.

Convenient notations Einstein summation convention is adopted. Furthermore, partial differentiation is sometimes denoted by $V_{,i} := \partial_i V := \partial V(x)/\partial x^i$.

2 Calculus on manifolds

To understand the BV formalism, we begin by reviewing differential geometry and algebraic tools—forms, vector fields, etc.—because they are the language in which classical and quantum field theories are expressed.

2.1 Ingredients of calculus

The following are main characters of calculus, which we are going to review one by one: Smooth space, function algebra, vector fields, polyvector fields, Lie bracket, differential forms, de Rham differential, pairing between forms and fields, and Lie derivative.

2.1.1 Smooth manifold

A topological manifold M is a topological space locally modeled on Euclidean spaces $\mathbb{R}^n$. This means that

1. We have an open cover $\{U_\alpha\}$ of M,
2. homeomorphisms ('local charts') $\phi_\alpha : U_\alpha \to \mathbb{R}^n$,
3. such that the transition functions $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ are homeomorphisms.

If the transition functions are in fact diffeomorphisms, we can regard M as an object in which we can do virtually every operation (e.g. differentiation of a function) in calculus. This is done by declaring that the local charts ϕ_α are diffeomorphisms. Many notions of vector calculus in $\mathbb{R}^n$ can be transported to M (then called *smooth manifold*) in such a way.

2.1.2 Function algebra

Calculus on Euclidean spaces $\mathbb{R}^n$ is the study of smooth (=infinitely differentiable) functions $f(x^1, x^2, \cdots, x^n) \in C^\infty(\mathbb{R}^n)$. In particular, the coordinate functions x^i are smooth.

Following the strategy described previously, we define a smooth function $f : M \to \mathbb{R}$ as a collection of functions

$$f_\alpha : U_\alpha \to \mathbb{R} \tag{1}$$

where each $f_\alpha \circ \phi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}$ is smooth and the consistency condition is met,

$$f_\alpha\big|_{U_\alpha \cap U_\beta} = f_\beta\big|_{U_\alpha \cap U_\beta}, \tag{2}$$

so that the functions f_α can be glued together to form a single function on M. In principle, the same strategy applies for every global object that can be defined on a manifold (e.g. vector field, bundle, etc.).

2.1.3 Vector fields

In introductory calculus, the variation of a function $f(x)$ under infinitesimal variations of the point x is measured by the directional derivative, which is a function defined on arbitrary vectors (directions).

$$v \in \mathbb{R}^n \;\mapsto\; (\nabla_v f)(x) := \frac{d}{dt}\bigg|_{t=0} f(x + tv) = v^i \frac{\partial f}{\partial x^i}(x) \in \mathbb{R} \tag{3}$$

On the other hand, as an operator on $C^\infty(\mathbb{R}^n)$, directional derivatives ∇_v are linear operators that satisfy the Leibniz rule.

$$\nabla_v : C^\infty(\mathbb{R}^n) \to \mathbb{R} \tag{4a}$$
$$\nabla_v(c_1 f + c_2 g) = c_1 \nabla_v f + c_2 \nabla_v g, \tag{4b}$$
$$\nabla_v(fg) = (\nabla_v f)g + f\nabla_v g. \tag{4c}$$

Given a vector field $X : \mathbb{R}^n \to \mathbb{R}^n$, the directional derivatives of f given by X collectively form a smooth function $\nabla_X f \in C^\infty(\mathbb{R}^n)$. The operator $\nabla_X : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is again linear and Leibniz.

More generally, given a $\mathbb{R}$-algebra A and a $\mathbb{R}$-vector space M, we define the class $\mathrm{Der}_{\mathbb{R}}(A, M)$ of operators by

$$X \in \mathrm{Der}_{\mathbb{R}}(A, M) \quad \Leftrightarrow \quad \begin{array}{c} X(c_1 f + c_2 g) = c_1 Xf + c_2 Xg, \\ X(fg) = (Xf)g + fXg, \\ \forall c_1, c_2 \in \mathbb{R}, \quad \forall f, g \in A. \end{array} \tag{5}$$

The class $\mathrm{Der}_{\mathbb{R}}(A, M)$ is called the $\mathbb{R}$-*derivations* of A with value M. It turns out that the vectors and vector fields on $\mathbb{R}^n$ are naturally identified with certain derivations.

$$\{\text{vectors}\} \quad \leftrightarrow \quad \mathrm{Der}_{\mathbb{R}}(C^\infty(\mathbb{R}^n), \mathbb{R}), \tag{6a}$$
$$\{\text{vector fields}\} \quad \leftrightarrow \quad \mathrm{Der}_{\mathbb{R}}(C^\infty(\mathbb{R}^n), C^\infty(\mathbb{R}^n)) \tag{6b}$$

This can be reversed to *define* vector (field) by (6). The advantage is that we have a coordinate-free definition of vector fields on curved space M.

$$\mathfrak{X}(M) := \mathrm{Der}_{\mathbb{R}}(C^\infty(M), C^\infty(M)) \tag{7}$$

2.1.4 Lie bracket

Given a pair of vector fields $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^i \frac{\partial}{\partial x^i}$, one can check that the Lie bracket $[X, Y]$ locally given by

$$[X, Y] = \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j} \tag{8}$$

is a vector field. Globally, this follows from the algebraic fact that

$$X, Y \in \mathrm{Der}_k(R, R) \quad \rightsquigarrow \quad [X, Y] := XY - YX \in \mathrm{Der}_k(R, R) \tag{9}$$

where R is a k-algebra. Since we can define vector fields on a curved space M as derivations, Lie bracket is globally well-defined.

2.1.5 Differential 1-forms

By definition, 1-forms are dual to vector fields;

$$\Omega^1(M) := \mathrm{Hom}_{C^\infty(M)}(\mathfrak{X}(M), C^\infty(M)). \tag{10}$$

In a Euclidean space (or the coordinate charts of a manifold), the duals dx^i to the elementary vector fields $\partial/\partial x^i$ form a basis of $\Omega^1(M)$.

$$dx^i \left(\frac{\partial}{\partial x^j} \right) = \delta^i_j \tag{11}$$

2.1.6 Differential k-forms

The k-forms are defined as the exterior products of 1-forms.

$$\Omega^k(M) := \bigwedge^k \Omega^1(M) \tag{12}$$

The exterior power $\bigwedge^k V$ encodes the skew-symmetry,

$$\bigwedge^k V := \frac{\bigotimes^k V}{\left\langle v_1 \otimes \cdots \otimes v_k - (-1)^{|\sigma|} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \right\rangle_{\sigma \in S_k}}. \tag{13}$$

The skew-symmetry is an algebraic consequence of imposing $\alpha^2 = 0$ for all $\alpha \in \Omega^1(M)$ ("degenerate parallelpiped has zero area"). In other words, as long as we want higher forms to encode volume element in M, skew-symmetry is inevitable.

2.1.7 de Rham differential

The de Rham differential is a degree-shifting operator on differential forms, which is uniquely determined by the following (local) properties.

$$df = \frac{\partial f}{\partial x^i} dx^i, \qquad d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{|\omega||\eta|} \omega \wedge d\eta. \tag{14}$$

It is illustrated in $\mathbb{R}^3$ as

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz, \tag{15a}$$

$$d(f dx + g dy + h dz) \tag{15b}$$

On a curved space, the de Rham differentials in coordinate charts glue well into a global operator. As can be seen from the local definition, it squares to zero, and endow $\Omega^\bullet(M)$ with a cochain complex structure:

$$d : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M), \qquad d^2 = 0. \tag{16}$$

Note that skew-symmetry enforces $\Omega^k(M) = 0$ for $k > n$, where n is the dimension of space.

There is a natural pairing between fields on one hand, which are relevant to differentiation, and forms on the other hand, which are relevant to integration.

2.1.8 Interior product, or "pairing"

Since 1-forms are defined as the duals of vector fields, we have a natural evaluation map

$$i : \mathfrak{X}(M) \otimes \Omega^1(M) \to C^\infty(M). \tag{17}$$

More generally, we define the interior product by extending the evaluation skew-symmetrically.

$$i : \mathfrak{X} \otimes \Omega^\bullet \to \Omega^{\bullet-1}$$

$$(X, \omega) \mapsto \sum_{i=1}^n (-1)^n \omega_{\mu_1\mu_2\cdots\mu_n} X^i dx^{\mu_1} \wedge \cdots d\hat{x}^{\mu_i} \cdots \wedge dx^{\mu_n} \tag{18}$$

2.1.9 Polyvector field

The pairing of forms and fields suggest that it is natural to extend vector fields, as well as forms, into anti-symmetric tensors. They are called multivectors or polyvectors.

$$P^k(M) := \bigwedge^k \mathfrak{X}(M) \tag{19}$$

Then the full-fledged interior product is the pairing

$$i : P^i \otimes \Omega^j \to \Omega^{j-i} \tag{20}$$

2.1.10 Lie derivative

Finally, the Lie derivative is a degree 1 operation that satisfies the Leibniz rule.

$$\begin{aligned} L : \mathfrak{X} \otimes \Omega^1 &\longrightarrow \Omega^1 \\ X \otimes \alpha &\longmapsto d(\alpha(X)) \end{aligned} \tag{21}$$

In general, Lie derivative is extended to a map $L : \mathfrak{X} \otimes \Omega^j \to \Omega^j$ using antisymmetry and Leibniz rule.

2.2 Cartan calculus

We notice that the fundamental operations of calculus satisfy many compatibility relations:

Proposition 2.1. *Let i, L, d be the interior product, Lie derivative, and de Rham differential on a manifold M. Then the following hold.*

1. $i_{X \wedge Y} = i_X \circ i_Y$
2. $[i_X, L_Y] = i_{[X,Y]}$
3. $[L_X, L_Y] = L_{[X,Y]}$
4. $[L_X, d] = 0$
5. $[i_X, d] = L_X$

The last 'commutator' is in fact anti-commutator since the degrees of i_X, d are both odd.

Definition 2.2 (Gerstenhaber algebra)**.** *It is a tuple $(P^\bullet, \cdot, \{-,-\})$ of graded vector space, graded-commutative multiplication, and shifted Poisson bracket.*

$$\cdot : P^\bullet \otimes P^\bullet \to P^\bullet, \qquad \{-,-\} : P^\bullet \otimes P^\bullet \to P^\bullet \tag{22}$$

1. $X \cdot Y = (-1)^{|X||Y|} Y \cdot X$
2. $\{X, Y\} = -(-1)^{(|X|-1)(|Y|-1)}\{Y, X\}$
3. $\{X, \{Y, Z\}\} = \{\{X, Y\}, Z\} + (-1)^{(|X|-1)(|Y|-1)}\{Y, \{X, Z\}\}$
4. $\{X, Y \cdot Z\} = \{X, Y\}Z + (-1)^{(|X|-1)|Y|} Y \cdot \{X, Z\}$

Secretly we regard the degree 1 part of Gerstenhaber algebra as vector fields on a space: $P^1 = \mathfrak{X}(M)$.

Example 2.3. *The polyvector fields $(P^\bullet(M) := \bigwedge^\bullet \mathfrak{X}(M), \wedge, [-,-])$ is a Gerstenhaber algebra.*

Remark 2.4. *The degree of the internal product*

$$i : P^i \otimes \Omega^j \to \Omega^{j-i} \tag{23}$$

is defined to be 0. This suggests that the grading of $\Omega^\bullet$ is in the reverse direction with respect to $P^\bullet$, which has a motivation in physics.

Definition 2.5 (Cartan calculus)**.** *A calculus is a tuple $((P^\bullet, \cdot, \{-,-\}), (\Omega^\bullet, d), i, L)$ of*

1. *Gerstenhaber algebra $(P^\bullet, \cdot, \{-,-\})$,*
2. *cochain complex $(\Omega^\bullet, d)$,*
3. *actions $i, L : P^\bullet \circlearrowright \Omega^\bullet$ of degree 0 and -1, respectively, satisfying the compatibility conditions.*

Remark 2.6. *Why not consider wedge product of forms? Ans: it is not a fundamental ingredient and only arises in commutative spaces*

2.3 Integration of forms

$$\Omega^0 \to \Omega^1 \to \cdots \to \Omega^n \tag{24}$$

$\Sigma \in C_k(M)$ from chain complex $\rightsquigarrow \int_\Sigma : \Omega^k \to \mathbb{R}$

Theorem 2.7 (Stokes)**.** *Given a $(k-1)$-form $\alpha \in \Omega^{k-1}$ and a k-dimensional submanifold Σ,*

$$\int_\Sigma d\alpha = \int_{\partial\Sigma} \alpha \tag{25}$$

Corollary 2.8. *Integration of forms has following properties.*

- *If α is exact ($\alpha \in \operatorname{im} d$) and Σ is closed ($\partial\Sigma = \emptyset$), then $\int_\Sigma \alpha = 0$.*
- *If α is closed ($\alpha \in \ker d$) and two submanifolds Σ_1, Σ_2 are homologous (i.e. they are representatives of a same cohomology class $\sigma = [\Sigma_1] = [\Sigma_2] \in H^*(M)$), then $\int_{\Sigma_1} \alpha = \int_{\Sigma_2} \alpha$.*
- *As a result, the pairing $\int_{(-)}(-) : H^*(M) \otimes H(M) \to \mathbb{R}$ is well-defined.*

2.3.1 Volume form

$dV = \rho(x) dx^1 \wedge \cdots \wedge dx^n \in \Omega^n(M)$

$$\int_M dV : C^\infty(M) \xrightarrow{\cdot dV} \Omega^n(M) \xrightarrow{\int_M} \mathbb{R} \tag{26}$$

Recall $i : P^i \otimes \Omega^j \to \Omega^{j-i}$. In particular, the first map in the above is a special case $i : P^0 \otimes \Omega^n \to \Omega^n$.

More generally, $i_{(-)}dV$ establishes a duality between $\Omega^\bullet \xrightarrow{\sim} P^{n-\bullet}$, and induces a differential Δ on $P^\bullet$. It is called "BV-Laplacian".

$$\begin{tikzcd}
\Omega^0(M) \arrow[r, "d"] & \Omega^1(M) \arrow[r, "d"] & \cdots \arrow[r, "d"] & \Omega^{n-1}(M) \arrow[r, "d"] & \Omega^n(M) \\
P^n(M) \arrow[r, "\Delta"] \arrow[u, "i_{(-)}dV"] & P^{n-1}(M) \arrow[r, "\Delta"] \arrow[u, "i_{(-)}dV"] & \cdots \arrow[r, "\Delta"] & P^1(M) \arrow[r, "\Delta"] \arrow[u, "i_{(-)}dV"] & P^0(M) \arrow[u, "i_{(-)}dV"]
\end{tikzcd} \tag{27}$$

The familiar divergence, curl, and gradient operations in $\mathbb{R}^3$ are special cases of BV-Laplacian.

$$\begin{tikzcd} \Omega^0(\mathbb{R}^3) \arrow[r, "d"] & \Omega^1(\mathbb{R}^3) \arrow[r, "d"] & \Omega^2(\mathbb{R}^3) \arrow[r, "d"] & \Omega^3(\mathbb{R}^3) \\ C^\infty(\mathbb{R}^3) \arrow[u] \arrow[r, "\mathrm{grad}"] & \mathfrak{X}(\mathbb{R}^3) \arrow[u] \arrow[r, "\mathrm{curl}"] & \mathfrak{X}(\mathbb{R}^3) \arrow[u] \arrow[r, "\mathrm{div}"] & C^\infty(\mathbb{R}^3) \arrow[u] \end{tikzcd} \tag{28}$$

The existence of an isomorphism $i_{(-)}dV$ indicates that bookkeeping $\Omega^\bullet$ and $P^\bullet$ together is highly redundant. Instead, one can completely forget about the forms $\Omega^\bullet$ after fetching the information of de Rham differentials via the volume form. $P^\bullet$ is then equipped with

$$\Delta = [i_{(-)}dV]^{-1} \circ d \circ [i_{(-)}dV]. \tag{29}$$

Then the axioms of Cartan calculus imply that

$$(X, Y) := \Delta(X \cdot Y) - (\Delta X) \cdot Y - (-1)^{|X|} X \cdot \Delta Y \tag{30}$$

becomes a shifted Poisson bracket. This suggests the following definition.

Definition 2.9. *A BV-algebra is a tuple* $(P^\bullet, \cdot, \{-,-\}, \Delta)$ *of*

1. *Gerstenhaber algebra* $(P^\bullet, \cdot, \{-,-\})$*,*
2. *degree* -1 *operator* Δ *such that* $\Delta^2 = 0$*,*

such that $\Delta(X \cdot Y) - (\Delta X) \cdot Y - (-1)^{|X|} X \cdot \Delta Y = \{X, Y\}$*.*

Philosophically, a BV-algebra is a (Cartan) calculus with a volume form. Again, Δ is called a BV-Laplacian. The bracket of a BV-algebra measures the failure of Δ being a derivation.

Remark 2.10. *A differential operator* $\mathrm{D} : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ *is of second order, if and only if,* $\{f, g\}_{\mathrm{D}} := \mathrm{D}(fg) - \mathrm{D}(f) \cdot g - f \cdot \mathrm{D}(g)$ *defines a Lie bracket. It is a motivation behind the name "BV-Laplacian".*

Definition 2.11. *Let B be a BV-algebra. An integration over B is a cocycle* $I : (B, \Delta) \to k$*. That is,* $I \circ \Delta = 0$*.*

3 Interlude: Mechanics of particles and fields

In this section, we review the field theory as a mathematical machine that takes in certain geometric data and outputs a set of quantities as a function of the input. A field theory is identified as classical or quantum depending on the type of its output. The particle mechanics is a special kind of field theory, whose spacetime dimension is 1.

Definition 3.1. *A d-dimensional field theory consists of the following input data.*

1. *d-dimensional smooth manifold (spacetime)*
2. *Space of sections of a fiber bundle:* $\mathcal{F} = \Gamma(E \to M)$ *(space of fields)*
3. *A functional* $S : \mathcal{F} \to \mathbb{C}$ *(action)*

Classical output:

$$\mathcal{EOM} := \mathrm{Crit}(S) = \{\varphi \in \mathcal{F} : dS|_\varphi = 0\} \tag{31}$$

Quantum output:

$$\mathcal{Z}(M) := \int_{\mathcal{F}} e^{iS/\hbar} \mathcal{D}\varphi \qquad \textit{(partition function)} \tag{32a}$$

$$\langle O(\varphi) \rangle := \frac{1}{\mathcal{Z}(M)} \int_{\mathcal{F}} O(\varphi) e^{iS/\hbar} \mathcal{D}\varphi \qquad \textit{(correlation functions)} \tag{32b}$$

Equations (31) and (32) are not meant to be taken as mathematically rigorous expressions. As we will see, establishing nooks and crannies of these equations pose serious analytic problems. In fact, it is the goal of this note to provide an *algebraic* foundation of these pseudo-definitions using the BV formalism.

3.1 Classical mechanics: Particles vs. fields

3.1.1 Classical particles

A system of interest in classical mechanics is the pair (M, S) of a manifold M and an *action functional* S,

$$S : \mathrm{Map}([0,1], M) \longrightarrow \mathbb{R} \tag{33}$$

which is often given by an integral of local functional $\mathcal{L}$, called the *Lagrangian*.

$$S[\gamma] = \int_0^1 \mathcal{L}(\gamma(t), \dot\gamma(t), \ddot\gamma(t), \ldots, \gamma^{(k)}(t))dt \tag{34}$$

An action S determines the particle motion in the following way. Denoting the mapping space by

$$\mathcal{M} = \mathrm{Map}([0,1], M), \tag{35}$$

its cotangent bundle $T^*\mathcal{M}$ can be constructed as

$$T^*_\gamma\mathcal{M} = \Gamma([0,1], \gamma^*T^*M), \qquad T^*\mathcal{M} = \bigcup_\gamma T^*_\gamma\mathcal{M} \subset \mathrm{Map}([0,1], T^*M). \tag{36}$$

There are two natural sections of the cotangent bundle determined by the mechanical system (M, S). Namely, the zero section and the graph of dS.

$$\mathcal{M} \xrightarrow[dS]{0} T^*\mathcal{M} \tag{37}$$

The physical trajectory of a particle, or its *equation of motion*, is given by the intersection of two Lagrangian submanifolds

$$\mathcal{EOM} = \mathrm{Crit}(S) = \mathcal{M} \cap \mathrm{graph}(dS) \subset T^*\mathcal{M}. \tag{38}$$

Example 3.2 (particle on a manifold)**.** *Given a Riemannian manifold (M, g) together with a smooth function $V : M \to \mathbb{R}$ called 'potential', the motion of a particle on M is determined by the Lagrangian*

$$\mathcal{L}(x, \dot x) = \frac{1}{2}\langle \dot x, \dot x\rangle_g - V(x). \tag{39}$$

In particular, if $V = 0$, the particle motion follows a geodesic of M.

3.1.2 Classical fields

Typical spaces of fields are

$$\mathcal{F} = \mathrm{Map}(\Sigma, M) \quad \text{or} \quad \Gamma(\Sigma, E), \tag{40}$$

and we formally consider the intersection of two Lagrangian subspaces of the cotangent bundle $T^*\mathcal{F}$.

$$\mathcal{F} \xrightarrow[dS]{0} T^*\mathcal{F} \tag{41}$$

However, actually realizing this idea is considerably harder than the case of particle mechanics.

Why is field theory harder than particle mechanics?

While both classical particle mechanics and field theory can be formulated as variational problems over infinite-dimensional configuration spaces, the analytic challenges involved are vastly different in scale and nature. In this section, we clarify why field theory requires a far more delicate functional-analytic foundation, convincing the reader of the advantage of an algebraic approach as taken by the Batalin–Vilkovisky (BV) formalism.

For both particles and fields, the space of configurations can be viewed as a mapping space:

- For particle mechanics: configurations are paths $\gamma : [0,1] \to M$, so the configuration space is $\mathrm{Map}([0,1], M)$.
- For field theory: configurations are fields $\phi : \Sigma \to V$, giving $\mathrm{Map}(\Sigma, V)$, where Σ is a manifold of dimension $d > 1$.

In both cases, the action functional S is a function on this infinite-dimensional space, and its differential dS is formally a section of the cotangent bundle. However, the nature of the domain Σ and the required regularity of ϕ make a crucial difference in what the mapping space is ought be be.

Dimensionality and Compactness. The core analytic difficulty arises from the difference in domain:

- In mechanics, the domain is a compact interval $[0,1] \subset \mathbb{R}$, making most variational problems manageable with basic tools like $C^2([0,1])$, Sobolev space $H^2([0,1])$, or even smooth functions.
- In field theory, the domain Σ is typically higher-dimensional and noncompact, e.g., $\mathbb{R}^d$. To ensure that the action functional and its variational derivatives are well-defined, one must work with carefully chosen Sobolev spaces $H^s(\Sigma)$, often requiring large s to impose regularity.

Functional Derivatives and Distributions. In field theory, variational derivatives often take the form of distributions:

$$\frac{\delta S}{\delta \phi(x)} \in \mathcal{D}'(\Sigma),$$

and may involve ill-defined operations such as products of distributions or pointwise delta functions. Even after posing and solving the variational problem, the resulting equation of motion will be a partial differential equation whose solution quickly deviates from being smooth even when the initial condition is smooth.

Finiteness of Interactions. The space of fields must be carefully chosen to guarantee well-posedness of the action and its differential. For instance:

$$\phi \in H^s_{\mathrm{loc}}(\mathbb{R}^4), \quad s > 1,$$

ensures that $\phi^4 \in L^1_{\mathrm{loc}}$ and the action $S[\phi] = \int \phi^4 \, dx$ is finite; see Sobolev embedding theorem. This type of constraint is far more subtle than in the 1D case.

Gauge Redundancy and Quotients. In many physically relevant field theories, the configuration space carries a gauge symmetry, so the physical field space is a quotient:

$$\mathcal{F}_{\mathrm{phys}} = \mathcal{F}/\mathcal{G},$$

where $\mathcal{G}$ is an infinite-dimensional gauge group (often a Fréchet–Lie group). Such quotients are rarely smooth manifolds and require the language of derived geometry to be treated rigorously — precisely the motivation behind the BV formalism.

Conclusion. Although both particle mechanics and field theory appear to live on mapping spaces, the analytic complexity of the latter is considerably greater.

3.2 Path integral: Particles vs. fields

In both particle mechanics and quantum field theory, path integrals involve integrating over spaces of field configurations that are typically infinite-dimensional. These field configurations are almost never smooth in the functional integral formulation, and it is precisely this fact that demands careful analytic treatment. However, the analytic complexity of this situation differs significantly between particle mechanics and field theory.

3.2.1 Quantum particles

Consider the configuration space of a point particle moving in a manifold M, represented as a map $x : [0,1] \to M$. There is a procedure called *Wick rotation*, which replaces the path integral into its *Euclidean* counterpart.

takes the form:

$$\mathcal{Z} = \int \mathcal{D}x(t)\, e^{-S_E[x]/\hbar},$$

and the action $S_E[x]$ typically involves expressions like

$$S_E[x] = \int_0^1 \left(\frac{1}{2} m\dot{x}^2(t) + V(x(t)) \right) dt.$$

While the classical path $x_{\text{cl}}(t)$ is smooth, the quantum path integral includes contributions from highly non-smooth paths. Nonetheless, since the domain is one-dimensional and the measure is Gaussian in the free case (i.e. when $V(x)$ is quadratic), the entire framework can be rigorously formulated using Wiener measure or L^2-based Sobolev spaces such as $H^1([0,1])$. These function spaces are well-behaved, and the measure-theoretic aspects are controllable.

The Wiener measure is obtained by "regulating" the path integral with discrete time,

$$P = \{0 = t_0 < t_1 < t_2 < \cdots < t_N = 1\}, \quad \begin{array}{rr} \Delta t_0 = t_0, & \Delta x_0 = x(t_0), \\ \Delta t_i = t_i - t_{i-1}, & \Delta x_i = x(t_i) - x(t_{i-1}), \end{array} \tag{42a}$$

$$C(A_0, A_1, \cdots, A_N) = \mathrm{ev}_{t_0}^{-1}(A_0) \cap \cdots \cap \mathrm{ev}_{t_N}^{-1}(A_N), \qquad \mathrm{ev}_{t_i}[x] = x(t_i), \tag{42b}$$

$$\mu_P(C(A_0, A_1, \cdots, A_N)) = \int_{A_0} dx_0 \cdots \int_{A_N} dx_N \prod_{i=0}^{N} \frac{1}{\sqrt{2\pi \Delta t_i}} \exp\left[-\frac{(\Delta x_i)^2}{2\Delta t_i} - V(x_i) \right]. \tag{42c}$$

and taking the limit $|P| = \max_i \Delta t_i \to 0$. Then the measure μ_P weakly converges to a well-defined *Gibbs measure*, aka *weighted Wiener measure.*

$$\mathcal{D}x(t)\, e^{-S_E[x]/\hbar} := \lim_{|P|\to 0} \mu_P \quad \text{(weak)}. \tag{43}$$

3.2.2 Quantum fields

In contrast, consider a scalar field theory on $\mathbb{R}^4$ with action:

$$S_E[\phi] = \int_{\mathbb{R}^4} \left(\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!}\phi^4 \right) dx.$$

The path integral formally reads:

$$\mathcal{Z} = \int \mathcal{D}\phi\, e^{-S_E[\phi]/\hbar},$$

but here, the configuration space is a space of functions $\phi : \mathbb{R}^4 \to \mathbb{R}$, often modeled by local Sobolev spaces $H^s_{\text{loc}}(\mathbb{R}^4)$ or even distribution spaces. These spaces contain functions that are not smooth, and indeed, the measure (even in the free Gaussian case) assigns measure zero to the space of smooth functions C^∞. Moreover, non-smooth configurations dominate the integral. However, in contrast to the one-dimensional case, Gaussian measures in higher dimensions require intricate renormalization and regularization procedures due to ultraviolet divergences.

Conclusion. While non-smooth field configurations are essential in both particle mechanics and field theory, the analytic landscape is far more forgiving in the former. In field theory, the combination of higher-dimensional domains, nonlinear interactions, and gauge redundancies make the definition of the path integral and its configuration space substantially more subtle. As a result, quantum field theory requires powerful techniques from functional analysis, Sobolev theory, and even derived geometry to make precise mathematical sense of its formal constructions.

3.3 Derived geometry motivation

In classical mechanics, the equation of motion is determined by the critical points of an action functional $S\colon \mathcal{M} \to \mathbb{R}$. Geometrically, this corresponds to the intersection

$$\mathrm{Crit}(S) = \mathcal{M} \cap \mathrm{graph}(dS) \subset T^*\mathcal{M},$$

where $\mathcal{M}$ is identified with the zero section of the cotangent bundle $T^*\mathcal{M}$, and $\mathrm{graph}(dS)$ is a Lagrangian submanifold given by the differential of the action. In typical mechanical systems, the well-posedness of the problem of finding $\mathrm{Crit}(S)$ is supported by a huge body of works in global analysis.

However, in field theory, this geometric picture becomes more subtle. The path integral formulation requires a rigorous formulation of integration over the infinite-dimensional field space $\mathcal{F} = \Gamma(E \to M)$, which is largely intractable in the current state of the art. Moreover, physical theories often come equipped with gauge symmetries. In such cases, the action functional $S\colon \mathcal{F} \to \mathbb{R}$ is invariant under infinitesimal transformations, and its differential dS vanishes along gauge orbits. This implies that $\mathrm{graph}(dS)$ and the zero section of $T^*\mathcal{F}$ badly fail to intersect transversely.

To correctly capture the geometry of this non-transverse intersection, algebraic geometry suggests to replace the classical critical locus by its *derived* counterpart. The derived critical locus is defined by the cohomology of the function algebra $\mathcal{O}(T^*[-1]\mathcal{F})$ of the *shifted cotangent bundle* of the field space, with respect to the differential $Q = \{S, -\}$:

$$\mathrm{Crit}(S) := \mathrm{Spec}\, H^\bullet(\mathcal{O}(T^*[-1]\mathcal{F}), Q).$$

This homological construction encodes both the equations of motion and the gauge symmetry data. It is this shift from classical to derived intersections that motivates the use of *graded manifolds* and the *BV formalism*, which will be developed in the following sections.

4 Geometry of graded manifolds

4.1 Graded vector spaces

Graded vector spaces are vector spaces which come with a decomposition into direct summands indexed by $\mathbb{Z}$. We have motivated their use by a brief discussion of field theory in the previous section. In fact, graded spaces are more than that, as various physical considerations suggest their importance, including the BRST as well as BV formalism and supersymmetry.

4.1.1 Definition and basic notions

Definition 4.1 (Graded vector space)**.** *A graded vector space $V = \oplus_{n\in\mathbb{Z}} V^n$ is a vector space decomposed into a direct sum of homogeneous components labeled by degrees. When $v \in V^n$ belongs to a homogeneous component, we denote by $|v| = n$ the degree of v.*

Example 4.2. *Main protagonists of our story are the graded vector spaces*

$$\Omega^\bullet(M): \text{ differential forms on } M, \quad P^\bullet(M): \text{ polyvector fields on } M. \tag{44}$$

Morphisms between graded vector spaces have natural grading. When V is a graded vector space, denote by $V[k]$ the k-shifted vector space defined as

$$V[k] = \oplus_{n\in\mathbb{Z}} V[k]^n, \qquad V[k]^n = V^{n+k}. \tag{45}$$

That is, $V[k]$ is identical to V as a plain vector space, but has different grading.

Definition 4.3. *A morphism $L : V \to W$ between graded vector spaces is just a linear map. Every linear map naturally decomposes into homogeneous components as follows.*

$$L = \bigoplus_{n\in\mathbb{Z}} L^n, \qquad L^n(V^k) \subseteq W^{n+k}. \tag{46}$$

That is, the degree n component of $L : V \to W$ is a map from V to $W[n]$.

$$L^n \in \mathrm{Hom}(V, W[n]) \tag{47}$$

4.1.2 Graded algebras

Koszul sign rule The Koszul sign rule governs how signs appear when permuting homogeneous elements in a graded context. Specifically, swapping two elements of degrees $|a|$ and $|b|$ introduces a sign $(-1)^{|a||b|}$. This rule is fundamental in ensuring the consistency of graded structures such as graded algebras, Lie brackets, and differential operators. We will review how this principle is incarnated in various graded vector spaces with additional structures.

Definition 4.4 (Symmetric algebra)**.** *The* <u>*symmetric algebra*</u> *of a graded vector space V is*

$$\mathrm{Sym}^\bullet(V) := \frac{\bigotimes^k V}{\left\langle v_1 \otimes \cdots \otimes v_k - (-1)^{\mathrm{Koszul}} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \right\rangle_{\sigma\in S_k}}. \tag{48}$$

The definition is analogous to (13), but the Koszul sign depends not only on the parity $|\sigma|$ of the permutation but also on the degrees of v_i. For pure elements that are the tensor product of two vectors in V, the Koszul sign is given by

$$v \otimes w = (-1)^{|v||w|} w \otimes v.$$

It follows that if V is a graded vector space concentrated in even degrees, then $\mathrm{Sym}^\bullet(V)$ coincides with the usual symmetric algebra. If V is in odd degrees, $\mathrm{Sym}^\bullet(V)$ becomes the exterior algebra, due to graded-commutativity.

Definition 4.5 (Graded Lie algebra)**.** *A* <u>*degree d graded Lie algebra*</u> *is a graded vector space L equipped with a bracket*

$$\{-,-\} : L \otimes L \to L[d] \tag{49}$$

that satisfies the following properties.

- *Graded antisymmetry:*
$$\{f, g\} = (-1)^{(|f|+d)(|g|+d)}\{g, f\}$$
- *Graded Jacobi identity:*
$$(-1)^{(|f|+d)(|h|+d)}\{f,\{g,h\}\} + (-1)^{(|g|+d)(|f|+d)}\{g,\{h,f\}\} + (-1)^{(|h|+d)(|g|+d)}\{h,\{f,g\}\} = 0$$

In a graded Lie algebra, Koszul signs behave as if the each side ('{' and '}') of the bracket carries a degree d. Whenever two elements f, g are exchanged, one should picture that the brackets move along with those elements; the Koszul sign records the total degree of "composite objects" exchanged ('$\{f$' and '$g\}$').

4.2 Graded spaces

Just as manifolds are formed by gluing vector spaces together, graded vector spaces can be globalized into graded manifolds.

4.2.1 Definition and basic notions

Slogan 4.6. *By a graded space we mean a ringed space $(X, \mathcal{O}_X)$ where the structure sheaf $\mathcal{O}_X$ is graded.*

Definition 4.7 (Graded manifold)**.** *A graded manifold is a pair $X = (M, \{E^n\}_{n\in\mathbb{Z}})$ of*

1. *smooth manifold M (called the body of X),*
2. *smooth vector bundles $E^n \to M$ parameterized by integers,*

together with the understanding that the function algebra of $X = (M, \{E^n\}_{n\in\mathbb{Z}})$ is given by

$$\mathcal{O}(X) := \bigoplus_{n\in\mathbb{Z}} \mathrm{Sym}^{\bullet}_{C^\infty(M)} \left(\Gamma(E^n)^\vee[-n]\right), \tag{50a}$$

$$\Gamma(E^n)^\vee := \mathrm{Hom}_{C^\infty(M)}(\Gamma(E^n), C^\infty(M)). \tag{50b}$$

We denote the graded manifold whose functions come from the vector bundles $\{E^n\}_{n\in\mathbb{Z}}$ by

$$X = \bigoplus_{n\in\mathbb{Z}} E^n[-n]. \tag{51}$$

Example 4.8 (Shifted tangent bundle)**.** *The shifted tangent bundle $T[1]M = (M, E^1 = TM)$ is defined by*

$$\begin{aligned}\mathcal{O}_{T[1]M} &= \mathrm{Sym}_{\mathcal{O}(M)}\, \Gamma(T[1]M)^\vee \\ &= \mathrm{Sym}_{\mathcal{O}(M)}(\mathfrak{X}(M)[1]^\vee) \\ &= \mathrm{Sym}_{\mathcal{O}(M)}(\Omega^1(M)[-1]) \\ &= \bigoplus_{k\in\mathbb{Z}} \wedge^k\Omega^1(M)[-k] \\ &= \bigwedge \Omega^1(M) \\ &= \Omega^\bullet(M)\end{aligned} \tag{52}$$

which means that the body of $T[1]M$ is M itself, and $\mathcal{O}_{T[1]M}(U) = \Omega^\bullet(U)$ for each open set $U \subseteq M$. In the first line of (52), we tacitly used the definition $\mathcal{O}_{T[1]M}(U) = \mathcal{O}_E(\pi^{-1}U)$. It follows from the definition of total space that this should be $\mathrm{Sym}_{\mathcal{O}(M)}\,\Gamma(T[1]M)^*$. The second part is the crucial step, where we fix what it means to shift a bundle. The remaining equalities are standard computations.

Denoting the elementary 1-forms $dx^1, \cdots dx^n$ in the coordinate system $(x^1, \cdots, x^n) \in U$ by $\theta^1, \cdots, \theta^n$, a typical function $f \in \mathcal{O}_{T[1]M}(U) = \Omega^\bullet(U)$ looks like

$$f(x,\theta) = g(x)\theta^{j_1} \wedge \cdots \wedge \theta^{j_d} \in \mathcal{O}_{T[1]M}(U)^d \tag{53}$$

Now, recall that vector fields can be defined as the derivations on the function algebra of a manifold:

$$\mathfrak{X}(X) := \mathrm{Der}(\mathcal{O}(X), \mathcal{O}(X)) \tag{54}$$

We extend this definition onto graded spaces $X = T[1]M$. Only using the directional vectors along the body M but not θ directions, we can write down a rather special vector field

$$Q := \sum_i \theta_i \frac{\partial}{\partial x^i}. \tag{55}$$

Given $f(x,\theta)$ as in (53),

$$\begin{aligned} Qf(x,\theta) &= g_{,i}\theta^i \wedge \theta^{j_1} \wedge \cdots \wedge \theta^{j_d} \\ &= \sum_i g_{,i} dx^i \wedge dx^1 \wedge \cdots \wedge dx^r \\ &= d(g(x)dx^1 \wedge \cdots \wedge dx^r) \end{aligned} \tag{56}$$

It follows that $Q^2 = 0$. Note that Q makes the function algebra a cochain complex. In this sense, $Q \in \mathfrak{X}(X)$ is called a cohomological vector field.

$$\mathcal{O}_X^0 \xrightarrow{Q} \mathcal{O}_X^1 \xrightarrow{Q} \mathcal{O}_X^2 \xrightarrow{Q} \cdots \tag{57}$$

There is a dual object of shifted tangent space.

Example 4.9 (shifted cotangent bundle)**.** *The* <u>*shifted cotangent bundle*</u> $T^*[-1]M$ *is a graded space given by* $X = (M, E^{-1} = T^*M)$*. Its function algebra is*

$$\begin{aligned} \mathcal{O}(X) &= \mathrm{Sym}\left(\Gamma(T^*M)^\vee[1]\right) \\ &= \bigwedge \mathfrak{X}(M)[1] \\ &= P^\bullet(M). \end{aligned} \tag{58}$$

4.2.2 Shifted symplectic manifold

Recall 4.10. *If* (X,ω) *is a symplectic manifold, then*

- $TX \simeq T^*X$.
- *The composition* $\mathcal{O}(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{\sim} \mathfrak{X}(X)$ *assigns a vector field* V_f *to each function* f.
- *The function algebra* $\mathcal{O}(X)$ *is endowed with a nontrivial Poisson bracket* $\{f,g\} = \omega(V_f, V_g)$.

Now we incorporate grading into these structures.

Definition 4.11 (forms on a graded space)**.** *Let* $(X, \mathcal{O}_X)$ *be a graded manifold. Using the tangent sheaf* TX *defined by*

$$\Gamma(U, TX) = \mathrm{Der}(\mathcal{O}_X(U), \mathcal{O}_X(U)), \tag{59}$$

the sheaf of differential forms $\Omega^\bullet(X)$ *is defined as*

$$\Omega^\bullet(X) := \mathcal{O}_{T[1]X} = \mathrm{Sym}^\bullet\, \Omega^1(X)[-1]. \tag{60}$$

Obviously, forms on a graded space are graded objects by themselves as they are decomposed into homogeneous components.

Definition 4.12. *A* <u>(-1)*-shifted symplectic form*</u> *on a graded manifold* X *is a sheaf of closed nondegenerate 2-form of homogeneous degree* -1

$$\omega : TX \otimes TX \to \mathcal{O}_X[-1]. \tag{61}$$

Proposition 4.13. *If the symplectic manifold* M *is graded, and the symplectic form* ω *is of degree* -1*, then* $(\mathcal{O}(X), \{-,-\})$ *becomes a* 1*-shifted Poisson algebra.*

Proof. The passage from symplectic form to Poisson bracket is fairly straightforward, so it will suffice to check the degree. A symplectic form ω induces a pair of "musical" isomorphism $(-)^\flat : TX \simeq T^*[-1]X : (-)^\sharp$ given by

$$V^\flat = \omega(V, -), \qquad \alpha = \omega(\alpha^\sharp, -). \tag{62}$$

In particular, if ω has degree k, then $(-)^\sharp$ operation has degree $-k$. It follows that the Poisson bracket

$$\{F, G\} = \omega(dF^\sharp, dG^\sharp) \tag{63}$$

carries degree $-k = 1$. □

Example 4.14. *Shifted cotangent bundle $X = T^*[-1]M$ is naturally equipped with a shifted symplectic form*

$$\omega = dx^i \wedge d\psi_i \tag{64}$$

where x^i is a coordinate chart on the body M and $\psi_i \in TM[1]$ is the vector $\partial/\partial x^i$ shifted to degree -1.

The (-1)-shifted symplectic form (64) induces a 1-shifted Poisson bracket on $\mathcal{O}(X)$, the polyvector fields. A nontrivial fact is that this bracket restricts to the usual commutator of vector fields in degree -1.

Proposition 4.15.

$$V, W \in \mathfrak{X}(M) \quad \Rightarrow \quad \{V, W\} = [V, W]. \tag{65}$$

What is precisely meant by Eq. (65) is the following. Since Poisson bracket takes two functions in its arguments, the vector fields V, W enclosed by the bracket are to be interpreted as the functions $V^i(x)\psi_i, W^j(x)\psi_j \in \mathcal{O}(X)$ on $X = T^*[-1]M$. Then the proposition asserts that

$$\{V^i(x)\psi_i, W^j(x)\psi_j\} = [V, W]^k(x)\psi_k = (V^i W^k_{,i} - W^j V^k_{,j})(x)\psi_k. \tag{66}$$

Proof. Using the musical isomorphism (62),

$$\alpha_V := d(V^i(x)\psi_i) = -V^i_{,j}(x)\psi_i dx^j + V^i(x)d\psi_i = \omega(\alpha_V^\sharp, -), \tag{67a}$$

$$\alpha_V^\sharp = V^i_{,j}\psi_i \frac{\partial}{\partial \psi_j} + V^i \frac{\partial}{\partial x^i}. \tag{67b}$$

Now the Poisson bracket is given by

$$\begin{aligned}
\{V^i(x)\psi_i, W^j(x)\psi_j\} &= \omega(\alpha_V^\sharp, \alpha_W^\sharp) \\
&= \omega\left(V^i_{,k}(x)\psi_i \frac{\partial}{\partial \psi_k} + V^i \frac{\partial}{\partial x^i},\ W^j_{,l}(x)\psi_j \frac{\partial}{\partial \psi_l} + W^j \frac{\partial}{\partial x^j}\right) \\
&= (V^i W^k_{,i} - W^j V^k_{,j})(x)\psi_k.
\end{aligned} \tag{68}$$

□

Hence, extending over the whole $\mathcal{O}(X) = P^\bullet(M)$,

$$\{X_1 \wedge \cdots \wedge X_n, Y_1 \wedge \cdots \wedge Y_m\} = \sum_{i,j} (-1)^{i+j}[X_i, Y_j] \wedge (X_1 \cdots \hat{X}_i \cdots X_n) \wedge (Y_1 \cdots \hat{Y}_j \cdots Y_m) \tag{69}$$

It is called the *Schouten bracket.*

Table 1 summarizes the calculus on shifted tangent and cotangent bundles.

X	$T[1]M$	$T^*[-1]M$
$\mathcal{O}(X)$	$\Omega^\bullet(M)$	$P^\bullet(M)$
Classical calculus	de Rham complex $(\Omega^\bullet(X), d)$	Shifted Poisson algebra of polyvector fields $(P^\bullet(X), \wedge, [-,-])$
Cartan calculus	A cochain complex $(\Omega^\bullet(X), Q)$	A Gerstenhaber algebra $(P^\bullet(X), \cdot, \{-,-\})$

Table 1: Summary of Sec. 4.2

4.3 Integration on graded manifolds

- $X = T[1]M \supseteq T[1]S$
- $\mathcal{O}(X) \ni f(x,\theta)$
- $\int_S : \Omega^\bullet(M) \to \mathbb{R} : g(x)dx^I \mapsto \int_S g(x)dx^I$, where $\dim S = |I| = k$

Considering $f(x,\theta) = g(x)\theta^I$, to wit:

Definition 4.16 (Berezin integral)**.**

$$\int_S \frac{\partial}{\partial \theta^{I(S)}} f(x,\theta)\Big|_{\theta=0} dx^{I(S)} =: \int_{T[1]S} f(x,\theta) \tag{70}$$

where $I(S)$ is the set of directional indices along S. This is called the Berezin integral.

Remark 4.17. *This is equivalent to the Grassmann calculus explained in physics textbooks. A Grassmann function $f(x,\theta) = f_0(x) + f_1(x)\theta$ of a single Grassmann variable θ has its own rules of differentiation and integration.*

$$\frac{\partial}{\partial\theta} f(x,\theta) = \int d\theta f(x,\theta) = f_1(x) \tag{71}$$

When there are several independent Grassmann variables ($\{\theta^i, \theta^j\} = 0$), their calculus is fixed by the additional rules

$$\int d\theta^i \theta^j = \frac{\partial}{\partial\theta^i}\theta^j = \delta^{ij}. \tag{72}$$

The Berezin integral, as in (70), is equivalent to

$$\int_{T[1]S} f(x,\theta) := \int_S dx^{I(S)} \int d\theta^{I(S)} f(x,\theta). \tag{73}$$

4.3.1 Volume form on graded manifold

Recall 4.18. *A volume form dV on a manifold M induces an isomorphism between polyvectors and forms:*

$$i_{(-)}dV : (P^\bullet(M), \Delta) \xrightarrow{\sim} (\Omega^\bullet(M), Q). \tag{74}$$

See (27).

Having established the calculus of graded manifolds, we are going to interpret the left-hand side of (74) as $\mathcal{O}(T^*[-1]M)$, and the right-hand side as $\mathcal{O}(T[1]M)$.

$$F_{dV} : \mathcal{O}(T^*[-1]M) \longrightarrow \mathcal{O}(T[1]M)$$

$$g(x,\psi) \longmapsto f(x,\theta) \tag{75}$$

Locally presenting the volume form as $dV = \rho(x)dx$, we observe that the contraction map $i_{(-)}dV$ replaces ψ_I by $\rho(x)\theta^J$, where $I \sqcup J = \{1,2,\ldots,n\}$. This property completely fixes the map F_{dV}, which is just the map $i_{(-)}dV$ together with grading information. To wit:

$$F_{dV}(g)(x,\theta) = \int d\psi e^{-\theta^i\psi_i} g(x,\psi)\rho(x) \tag{76}$$

where we used the notation of (71). (76) is called the *odd Fourier transform*. Using this, we can pull back the cohomological vector field $Q = d$ on $\Omega^\bullet$ to a BV-Laplacian on $P^\bullet$.

$$\begin{array}{ccc} P^{n-k}(M) & \xrightarrow{\Delta} & P^{n-k-1}(M) \\ \downarrow{\scriptstyle F_{dV}} & & \downarrow{\scriptstyle F_{dV}} \\ \Omega^{n+k}(M) & \xrightarrow{Q} & \Omega^{n+k+1}(M) \end{array} \tag{77}$$

Then the BV-Laplacian is given by

$$\Delta : \mathcal{O}(T^*[-1]M) \longrightarrow \mathcal{O}(T^*[-1]M)$$

$$\Delta g(x,\psi) = \frac{\partial^2}{\partial x^i \partial \psi_i} g(x,\psi) + \left[\frac{\partial}{\partial x^i} \log\rho(x)\right] \frac{\partial}{\partial \psi_i}\rho(x,\psi). \tag{78}$$

An interesting feature of odd Fourier transform is that it is well-behaved under restriction to subspaces. Given a submanifold $S \subseteq M$, we have the restriction

$$F_{dV}\big|_{N^*[-1]S} : \mathcal{O}(N^*[-1]S) \longrightarrow \mathcal{O}(T[1]S). \tag{79}$$

This is natural, since F_{dV} kills the ψ_I components in $N^*_x[-1]S \leq T^*_x[-1]M$ and plugs in θ^J with the complementary index set J. The directions in J have one-to-one correspondence with the directions in $T^*[-1]_xM/N^*_x[-1]S \simeq T_x[1]S$.

It turns out that the conormal bundle $N^*[-1]S$ over $S \subset M$ is a Lagrangian submanifold of $T^*[-1]M$. [Quick checking of half-dimensionality: $\dim S = k, \dim M = n \Rightarrow \dim N^*[-1]S = k + n - k = n = \frac{1}{2}\dim T^*[-1]M$] Even more, every Lagrangian submanifold of $T^*[-1]M$ is homologous to a conormal bundle. As a result, one can define an integral over the Lagrangian submanifolds of shifted cotangent bundle.

$$L \simeq N^*[-1]S \rightsquigarrow \int_L : \mathcal{O}(T^*[-1]M) \longrightarrow \mathbb{R}$$

$$\int_L g = \int_{N^*[-1]S} g(x,\psi)\rho(x)dxd\psi \tag{80}$$

Recipe: Odd Fourier transform followed by Berezin integral. Every such integration map $\int_L$ is a cocycle with respect to Δ.

Summary: Strategy of integral, or, the BV-quantization

We have established the following scheme.

$$\begin{gathered}\text{Shifted Poisson algebra } P\\ \downarrow\\ \text{Find an operator } \Delta : P \to P \text{ such that}\\ \Delta^2 = 0, \quad \Delta(xy) = \Delta(x)y + (-1)^{|x|}x\Delta(y) + \{x, y\}\\ \downarrow\\ \text{Find a cocycle } \int : (P, \Delta) \to k\\ \downarrow\\ \text{Get an integral } \int : P \to k\end{gathered} \tag{81}$$

5 BV formalism

5.1 Classical BV formalism

Recall that the output of a classical field theory with field space $\mathcal{F}$ and action S is the critical locus,

$$\mathrm{Crit}(S) = \mathcal{F} \times_{T^*\mathcal{F}} \mathcal{F}, \tag{82}$$

determined by the pair of maps $0, dS : \mathcal{F} \to T^*\mathcal{F}$. However, for the reason explained in Sec. 3.3, we would prefer replacing the ordinary critical locus by its derived counterpart. The resulting *derived critical locus* is realized by replacing the fiber product in (82) by the *derived fiber product*: $\mathrm{dCrit}(S) := \mathcal{F} \times^{\mathbb{R}}_{T^*\mathcal{F}} \mathcal{F}$.

$$\begin{tikzcd} \mathrm{Crit}(S) \arrow[d, equal] & \\ \mathcal{F} \times_{T^*\mathcal{F}} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, "0"] \\ \mathcal{F} \arrow[r, "dS"] & T^*\mathcal{F} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{dCrit}(S) \arrow[d, equal] & \\ \mathcal{F} \times^{\mathbb{R}}_{T^*\mathcal{F}} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, "0"] \\ \mathcal{F} \arrow[r, "dS"] & T^*\mathcal{F} \end{tikzcd} \tag{83}$$

On the function algebra level, it is equivalent to replacing the tensor product of algebras by the *derived tensor product*.

$$\begin{tikzcd} \mathcal{O}_{T^*\mathcal{F}} \arrow[r, "(dS)^*"] \arrow[d, "0"] & \mathcal{O}_{\mathcal{F}} \arrow[d] \\ \mathcal{O}_{\mathcal{F}} \arrow[r] & \mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{T^*\mathcal{F}}} \mathcal{O}_{\mathcal{F}} \arrow[d, equal] \\ & \mathcal{O}_{\mathrm{Crit}(S)} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}_{T^*\mathcal{F}} \arrow[r, "(dS)^*"] \arrow[d, "0"] & \mathcal{O}_{\mathcal{F}} \arrow[d] \\ \mathcal{O}_{\mathcal{F}} \arrow[r] & \mathcal{O}_{\mathcal{F}} \otimes^{\mathbb{L}}_{\mathcal{O}_{T^*\mathcal{F}}} \mathcal{O}_{\mathcal{F}} \arrow[d, equal] \\ & \mathcal{O}_{\mathrm{dCrit}(S)} \end{tikzcd} \tag{84}$$

In actual computations, the result of derived tensor product is determined by taking a *projective resolution* of an arrow, say $(dS)^*$, and then take the tensor product with the other arrow. The zeroth cohomology of the resulting chain complex may be taken for some purposes.

Algebraically, using the derived product resolves the non-transversal intersection of spaces by specifying the degeneracy data of the intersection points, the degeneracy data of degeneracy, and the degeneracy data of degeneracy of degeneracy, and so on.

We begin with a resolution of $\mathcal{O}_{\mathcal{F}}$, where $\mathcal{F}$ is understood to be the graph of dS in $T^*\mathcal{F}$. This means that $\mathcal{O}_{\mathcal{F}}$ is understood as a $\mathcal{O}_{T^*\mathcal{F}}$-algebra, where the vector V acts as

$$V \cdot f = V(S) f, \qquad V \in \mathcal{O}_{T^*\mathcal{F}}(U) \simeq \bigwedge \mathfrak{X}(\mathcal{F}). \tag{85}$$

Fortunately, there is a well-known resolution suitable for this situation:

$$\cdots \xrightarrow{Q} \mathcal{O}_{T^*\mathcal{F}} \underset{\mathcal{O}_{\mathcal{F}}}{\otimes} P^3_{\mathcal{F}} \xrightarrow{Q} \mathcal{O}_{T^*\mathcal{F}} \underset{\mathcal{O}_{\mathcal{F}}}{\otimes} P^2_{\mathcal{F}} \xrightarrow{Q} \mathcal{O}_{T^*\mathcal{F}} \underset{\mathcal{O}_{\mathcal{F}}}{\otimes} P^1_{\mathcal{F}} \xrightarrow{Q} \mathcal{O}_{T^*\mathcal{F}} \tag{86}$$

In (86), $\mathcal{O}_{T^*\mathcal{F}}$ is given a canonical $\mathcal{O}_{\mathcal{F}}$-algebra structure induced from the projection $T^*\mathcal{F} \to \mathcal{F}$. The differential Q acts on a vector $V \in \mathcal{P}_{\mathcal{F}}$ by $Q(V) = dS(V) = V(S)$.

$$Q(V) = V^i(x) Q\left(\frac{\partial}{\partial x^i}\right) = V^i(x)\frac{\partial S}{\partial x^i} = V(S) \tag{87}$$

Taking the tensor product with the $\mathcal{O}_{T^*\mathcal{F}}$-algebra $\mathcal{O}_{\mathcal{F}}$, whose module structure is induced by the zero section, one gets the cochain complex

$$\cdots \longrightarrow P^3_{\mathcal{F}} \xrightarrow{Q} P^2_{\mathcal{F}} \xrightarrow{Q} P^1_{\mathcal{F}} \xrightarrow{Q} \mathcal{O}_{\mathcal{F}} \tag{88}$$

of the $\mathcal{O}_{T^*\mathcal{F}}$-algebra $\mathcal{O}_{\mathcal{F}}$. Regarded as a *differential graded algebra*, (88) is precisely the function algebra of the shifted cotangent bundle over $\mathcal{F}$, together with the cohomological vector field Q. As promised in Sec. 3.3, this is precisely the function algebra of $T^*[-1]\mathcal{F}$! Using the notation in (64),

$$Q = \{S, -\} = \frac{\partial S}{\partial x^i}\frac{\partial}{\partial \psi_i}. \tag{89}$$

As a result, we reached

$$\begin{aligned} \mathcal{O}(\mathrm{dCrit}(S)) &= \mathcal{O}(\mathcal{F}) \otimes^{\mathbb{L}}_{\mathcal{O}(T^*\mathcal{F})} \mathcal{O}(\mathcal{F}) \\ &\simeq H^\bullet(P^\bullet(\mathcal{F}), Q = \{S, -\}) \\ &\simeq H^\bullet(\mathcal{O}(T^*[-1]\mathcal{F}), Q = \{S, -\}). \end{aligned} \tag{90}$$

The condition that Q is nilpotent imposes a constraint on the action functional. To this end, we use the the graded antisymmetry and graded Jacobi identity; see Def. 4.5. We observe that

$$\begin{aligned} Q^2(\alpha) &= \{S, \{S, \alpha\}\} \\ &= \{\{S, S\}, -\} + (-1)^{|\alpha|}\{\{S, \alpha\}, S\} \\ &= \{\{S, S\}, -\} - \{S, \{S, \alpha\}\}, \end{aligned} \tag{91a}$$

$$Q^2(\alpha) = \frac{1}{2}\{\{S, S\}, -\}, \tag{91b}$$

as S is a degree 0 function. Therefore, $Q^2 = 0$ if and only if $\{S, S\} = 0$. This is called the *classical master equation.*

Observation 5.1 (Classical master equation)**.** *The action functional S of a classical field theory satisfies $\{S, S\} = 0$.*

This leads to the following general definition.

Definition 5.2. *A classical BV system is a shifted Poisson algebra $(P^\bullet, \cdot, \{-, -\})$ together with $S \in P^0$ such that $\{S, S\} = 0$. Its equation of motion is given by*

$$\mathcal{EOM} = \mathrm{Spec}\, H^\bullet(P^\bullet, \{S, -\}). \tag{92}$$

5.2 Quantum BV formalism

BV-Laplacian associated to $dV = e^{iS/\hbar}\rho(x)dx$:

$$\Delta_S = \frac{\partial^2}{\partial x \partial \psi} + \left[\frac{\partial}{\partial x}\log\rho(x)\right]\frac{\partial}{\partial \psi} + \frac{i}{\hbar}\frac{\partial S}{\partial x}\frac{\partial}{\partial \psi} \tag{93}$$

Observe that

$$\frac{i}{\hbar}\frac{\partial S}{\partial x}\frac{\partial}{\partial \psi} = \frac{i}{\hbar}\{S, -\}. \tag{94}$$

In field theory, $\{S, S\} = 0$ is automatic since $S(x, \psi) \equiv S(x)$ does not depend on nonzero degree coordinates ψ. But we can also consider a more general functional S such that $\{S, S\} = 0$ (classical master equation). Suppose we have found a BV operator $\Delta : P \to P$.

$$\Delta_S : \Delta + \frac{i}{\hbar}\{S, -\} \tag{95}$$

Now, the path integral is defined as a cocycle w.r.t. Δ_S, i.e.

$$\int : P \to \mathbb{C} \quad \text{such that} \quad \int \Delta_S(-) = 0 \tag{96}$$

However, it remains to check that Δ_S is a BV-Laplacian. To this end, two comments are in order.

1. We need $\Delta_S^2 = 0$.
2. It turns out that $\Delta_S(xy) = \Delta_S(x)y + (-1)^{|x|}x\Delta_S(y) + \{x, y\}$ is automatically satisfied.

Just as $Q^2 = 0$ imposes a constraint on classical action, the requirement $\Delta_S^2 = 0$ constrains the quantum action S. It is called the *quantum master equation*.

Observation 5.3 (Quantum master equation)**.** $\Delta_S^2 = 0 \Leftrightarrow \hbar\Delta S + \frac{i}{2}\{S, S\} = 0$

Since $\hbar$ is a free variable, it is reasonable to expect that a solution S of quantum master equation depends on $\hbar$, possibly as a formal power series:

$$S \in \mathcal{O}(\mathcal{F})[[\hbar]]. \tag{97}$$

Definition 5.4 (BV-quantization)**.** *Let $(P, \cdot, \{-, -\}, S_0)$ be a classical BV system. Then a BV-quantization is an element $S = \sum_{n\geq 0} S_n \hbar^n \in P[[\hbar]]$ that solves the quantum master equation*

$$\hbar\Delta(S) + \frac{i}{2}\{S, S\} = 0. \tag{98}$$

The classical limit is obtained by taking $\hbar \to 0$, upon which the classical master equation $\{S_0, S_0\}$ is restored.

Unwinding the full quantum master equation, we get an infinite series of equations

$$\begin{aligned} \{S_0, S_0\} &= 0, \\ \hbar\Delta S_0 + \frac{i}{2}\left(\{S_1, S_0\} + \{S_0, S_1\}\right) &= 0, \\ \hbar\Delta S_1 + \frac{i}{2}\left(\{S_2, S_0\} + \{S_1, S_1\} + \{S_0, S_2\}\right) &= 0, \\ &\cdots \end{aligned} \tag{99}$$

which can be solved consecutively. After finding a solution S, the path integral is given by a cocycle

$$\int : (P[[\hbar]], \Delta_S) \to k[[\hbar]]. \tag{100}$$